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Applied Multi-Attribute Decision-Making with Complex Pythagorean Fuzzy Data based on Prioritized Aczel–Alsina Aggregation Operators: A Case for a Software Company

Tahira Karamat[1], Mehwish Sarfraz[1,*]

[1] Department of Mathematics, Riphah International University Lahore, (Lahore Campus) 5400, Lahore, Pakistan

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ABSTRACT

Complex Pythagorean fuzzy (C-PF) information is an extended form of fuzzy set theory. In this paper, we aim to compute the theory of prioritized aggregation operators based on Aczel-Alsina t-norm and t-conorm for managing the theory of C-PF information, such as C-PF prioritized Aczel-Alsina averaging, C-PF prioritized Aczel-Alsina ordered averaging, C-PF prioritized Aczel-Alsina geometric, and C-PF prioritized Aczel-Alsina ordered geometric operators. Three basic properties of the derived theory are also examined. Additionally, we illustrate the proposed multi-attribute decision-making technique for evaluating real-life problems related to software companies. Finally, we provide some examples to illustrate the contrast between derived work and previous or current information in order to demonstrate the ability and proficiency of the novel approach.

1. Introduction

The decision-making scenario is a capable procedure that is used for examining the finest preference from the collection of preferences, where the multiple-attribute decision-making (MADM) technique is the essential and critical part of the decision-making procedure. Furthermore, clustering analysis, pattern recognition, medical diagnosis, and image segmentation are also very famous and useful because of their demand, but in the presence of classical set theory, we faced a lot of problems because we have only two possibilities, such as zero or one. Because of the above complications, Zadeh [1] exposed or derived the mathematical and theoretical form of the fuzzy set (FS) by including a truth grade whose range is in the form of a unit interval. Furthermore, Atanassov [2] modified or extended the theory of FS and derived a new and valuable theory of intuitionistic FS (IFS) by including the term of truth and falsity grade with a law: $\mathbf{0} \le \mu^{\mathbf{1}}_{\dot{X}_{RP}}(\dot{w}) + \boldsymbol{\nu}^{\mathbf{1}}_{\dot{X}_{RP}}(\dot{w}) \le \mathbf{1}$. Beca-

* *Corresponding author.*
E-mail address: Mehwish.sarfraz.ms@gmail.com

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use of flaws in the existing idea of FS and IFS, the theory of Pythagorean FS (PFS) was derived by Yager [3] by modifying the law of IFS, such as $0 \leq \mu_{\ddot{X}_{RP}}^{2}(\dot{w}) + \nu_{\ddot{X}_{RP}}^{2}(\dot{w}) \leq 1$. After a successful investigation, various scholars have derived different types of information under the consideration of FS, IFS, and PFS theory, for instance, distance measures [4], correlation coefficient measures [5], divergence measures [6], and variance measures [7].

The phase term is also very famous and valuable because in many practical places we faced periodic functions which played a valuable and dominant role in the environment of real-life problems, for instance, when a person wants to buy a new branded type of the car, for this, he visits different car showrooms, the owner of each showroom give the type of data regarding each car such as name and production date of the car, which is represented the amplitude and phase term, in last existing theory, we discussed that the theory of FSs, IFSs, and PFSs have deal with one dimension information instead of two-dimension information, therefore, the main or major idea of complex FS (C-FS) was invented by Ramot *et al.* [8] by including the phase term in the term of truth grade whose range is in the form of a unit interval for each real part and imaginary part. Furthermore, Alkouri & Salleh [9] modified or extended the theory of C-FS and derived a new and valuable theory of complex IFS (C-IFS) by including the phase term in the field of truth and falsity grade with two same laws: $0 \leq \mu_{\ddot{X}_{RP}}^{1}(\dot{w}) + \nu_{\ddot{X}_{RP}}^{1}(\dot{w}) \leq 1$ and $0 \leq \mu_{\ddot{X}_{IP}}^{1}(\dot{w}) + \nu_{\ddot{X}_{IP}}^{1}(\dot{w}) \leq 1$. Because of flaws in the existing idea of C-FS and C-IFS, the theory of complex PFS (CPFS) was derived by Ullah *et al.* [10] by modifying the same laws of C-IFS, such as $0 \leq \mu_{\ddot{X}_{RP}}^{2}(\dot{w}) + \nu_{\ddot{X}_{RP}}^{2}(\dot{w}) \leq 1$ and $0 \leq \mu_{\ddot{X}_{IP}}^{2}(\dot{w}) + \nu_{\ddot{X}_{IP}}^{2}(\dot{w}) \leq 1$. After a successful investigation, various scholars have derived different types of information under the consideration of C-FS, C-IFS, and C-PFS theory, for instance, decision-making [11], group decision-making problems [12], Einstein aggregation operators (AOs) [13], and Archimedean AOs [14].

In the presence of the algebraic t-norm and t-conorm, the theory of prioritized aggregation operators (PAOs) under the consideration of priority degree, Yager [15] derived the theory of PAOs for classical information. Furthermore, Aczel-Alsina t-norm and t-conorm were invented by Alsina & Alsina [16] in 1982. Yu & Xu [17] exposed PAOs for IFSs. Senapati *et al.* [18] derived the AA AOs for IFSs. AA geometric AOs for IFS were developed by Senapati & [19]. PAOs for PFS were developed by Khan *et al.* [20]. AA AOs for PFS were invented by Senapati *et al.* [21], Hussain *et al.* [22], and Ul Haq *et al.* [23]. PAOs for C-IFS were invented by Garg & Rani [24]. AA AOs for C-IFS were exposed by Mahmood *et al.* [25]. Prioritized weighted AOs for C-PFS were derived by Akram *et al.* [26]. Finally, Jin *et al.* [27] exposed the theory of AA AOs for C-PF information and their application in decision-making.

C-PF information is the very famous and extended form of fuzzy set theory, which covers the grade of membership and the grade of non-membership with a valuable characteristic that is the sum of the squares of the duplet (for real and imaginary parts) should be contained in the unit interval. In this analysis. We aim to derive the following information, such as:

i. to compute the theory of the C-PF prioritized AA averaging (C-PFPAAA), C-PF prioritized AA ordered averaging (C-PFPAAOA), C-PF prioritized AA geometric (C-PFPAAG), and C-PF prioritized AA ordered geometric (C-PFPAAOG) operators;
ii. to examine the idempotency, monotonicity, and boundedness properties;
iii. to illustrate a MADM technique to describe the reliability and effectiveness of the derived theory;
iv. to demonstrate some numerical examples for showing the comparison between the derived work and existing approaches.

The main structure of this analysis is stated as: In section 2, we revised the idea of C-PF information and its operational laws. In Section 3, we computed the theory of C-PFPAAA, C-PFPAAOA, C-PFPAAG, and C-PFPAAOG operators. Further, we examined the idempotency, monotonicity, and boundedness properties. In Section 4, for evaluating various real-life problems, we illustrated an MADM technique based on the evaluated theory to describe the reliability and effectiveness of the derived theory. In Section 5, we demonstrated some numerical examples to show the comparison between the derived work and existing or prevailing information to state the capability and art of the derived approaches. In Section 6, we derived some concluding remarks.

2. Preliminaries

The main theme of this section is to revise the idea of C-PF information and its operational laws.

Definition 1 [10]**:** On a fixed set $\grave{W}$, we examined the C-PF set $\dot{X}$, such as

$$\dot{X} = \left\{\left(\left(\mu_{\dot{X}_{RP}}(\dot{w}), \mu_{\dot{X}_{IP}}(\dot{w})\right), \left(\nu_{\dot{X}_{RP}}(\dot{w}), \nu_{\dot{X}_{IP}}(\dot{w})\right)\right) : \dot{w} \in \grave{W}\right\} \tag{1}$$

where $\mu_{\dot{X}_{RP}}(\dot{w}), \mu_{\dot{X}_{IP}}(\dot{w}), \nu_{\dot{X}_{RP}}(\dot{w}), \nu_{\dot{X}_{IP}}(\dot{w}) \in [0,1]$ with $0 \leq \mu^2_{\dot{X}_{RP}}(\dot{w}) + \nu^2_{\dot{X}_{RP}}(\dot{w}) \leq 1$ and $0 \leq \mu^2_{\dot{X}_{IP}}(\dot{w}) + \nu^2_{\dot{X}_{IP}}(\dot{w}) \leq 1$. Moreover, $\hat{r}_{\dot{X}}(\dot{w}) = \left(\hat{r}_{\dot{X}_{PR}}(\dot{w}), \hat{r}_{\dot{X}_{IP}}(\dot{w})\right) = \left(\sqrt{1 - \left(\mu^2_{\dot{X}_{RP}}(\dot{w}) + \nu^2_{\dot{X}_{RP}}(\dot{w})\right)}, \sqrt{1 - \left(\mu^2_{\dot{X}_{IP}}(\dot{w}) + \nu^2_{\dot{X}_{IP}}(\dot{w})\right)}\right)$ represents the neutral grade where the simple form of PF value (PFV) is stated by: $\dot{X} = \left(\left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right), \left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2,..,n$.

Definition 2 [27]**:** Assume any two PFVs $\dot{X} = \left(\left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right), \left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2$, then we have

$$\dot{X}_1 \oplus \dot{X}_2 = \left(\begin{array}{c}\left(\sqrt{1 - e^{-\left(\left(-ln\left(1-\mu^2_{\dot{X}_{RP_1}}\right)\right)^{\daleth} + \left(-ln\left(1-\mu^2_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1 - e^{-\left(\left(-ln\left(1-\mu^2_{\dot{X}_{IP_1}}\right)\right)^{\daleth} + \left(-ln\left(1-\mu^2_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}\right), \\ \left(e^{-\left(\left(-ln\left(\nu_{\dot{X}_{RP_1}}\right)\right)^{\daleth} + \left(-ln\left(\nu_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(-ln\left(\nu_{\dot{X}_{IP_1}}\right)\right)^{\daleth} + \left(-ln\left(\nu_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right) \tag{2}$$

$$\dot{X}_1 \otimes \dot{X}_2 = \left(\begin{array}{c}\left(e^{-\left(\left(-ln\left(\mu_{\dot{X}_{RP_1}}\right)\right)^{\daleth} + \left(-ln\left(\mu_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(-ln\left(\mu_{\dot{X}_{IP_1}}\right)\right)^{\daleth} + \left(-ln\left(\mu_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right), \\ \left(\sqrt{1 - e^{-\left(\left(-ln\left(1-\nu^2_{\dot{X}_{RP_1}}\right)\right)^{\daleth} + \left(-ln\left(1-\nu^2_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1 - e^{-\left(\left(-ln\left(1-\nu^2_{\dot{X}_{IP_1}}\right)\right)^{\daleth} + \left(-ln\left(1-\nu^2_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}\right)\end{array}\right) \tag{3}$$

$$\varphi \dot{X} = \left(\begin{array}{c} \left(\sqrt{1 - e^{-\left(\varphi\left(-ln\left(1-\mu^2_{\dot{X}_{RP_1}}\right)\right)\right)^{ℸ}}}^{1/ℸ}, \sqrt{1 - e^{-\left(\varphi\left(-ln\left(1-\mu^2_{\dot{X}_{IP_1}}\right)\right)\right)^{ℸ}}}^{1/ℸ} \right), \\ \left(e^{-\left(\varphi\left(-ln\left(\nu_{\dot{X}_{RP_1}}\right)\right)\right)^{ℸ \ 1/ℸ}}, e^{-\left(\varphi\left(-ln\left(\nu_{\dot{X}_{IP_1}}\right)\right)\right)^{ℸ \ 1/ℸ}} \right) \end{array} \right) \quad (4)$$

$$\dot{X}^{\varphi} = \left(\begin{array}{c} \left(e^{-\left(\varphi\left(-ln\left(\mu_{\dot{X}_{RP_1}}\right)\right)\right)^{ℸ \ 1/ℸ}}, e^{-\left(\varphi\left(-ln\left(\mu_{\dot{X}_{IP_1}}\right)\right)\right)^{ℸ \ 1/ℸ}} \right), \\ \left(\sqrt{1 - e^{-\left(\varphi\left(-ln\left(1-\nu^2_{\dot{X}_{RP_1}}\right)\right)\right)^{ℸ}}}^{1/ℸ}, \sqrt{1 - e^{-\left(\varphi\left(-ln\left(1-\nu^2_{\dot{X}_{IP_1}}\right)\right)\right)^{ℸ}}}^{1/ℸ} \right) \end{array} \right) \quad (5)$$

Definition 3 [27]**:** Assume any two PFVs $\dot{X}_j = \left(\left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right), \left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2$. Then, we have

$$\acute{S}(\dot{X}) = \frac{1}{2}\left(\left(\mu^2_{\dot{X}_{RP_j}} + \mu^2_{\dot{X}_{IP_j}}\right) - \left(\nu^2_{\dot{X}_{RP_j}} + \nu^2_{\dot{X}_{IP_j}}\right)\right) \in [-1,1] \quad (6)$$

$$\acute{A}(\dot{X}) = \frac{1}{2}\left(\left(\mu^2_{\dot{X}_{RP_j}} + \mu^2_{\dot{X}_{IP_j}}\right) + \left(\nu^2_{\dot{X}_{RP_j}} + \nu^2_{\dot{X}_{IP_j}}\right)\right) \in [0,1] \quad (7)$$

Notice that:

i. If $\acute{S}(\dot{X}) < \acute{S}(\dot{Y}) \Rightarrow \dot{X} < \dot{Y}$;
ii. If $\acute{S}(\dot{X}) > \acute{S}(\dot{Y}) \Rightarrow \dot{X} > \dot{Y}$;
iii. If $\acute{S}(\dot{X}) = \acute{S}(\dot{Y})$, then If $\acute{A}(\dot{X}) > \acute{A}(\dot{Y}) \Rightarrow \dot{X} > \dot{Y}$ or If $\acute{A}(\dot{X}) < \acute{A}(\dot{Y}) \Rightarrow \dot{X} > \dot{Y}$ or If $\acute{A}(\dot{X}) = \acute{A}(\dot{Y}) \Rightarrow \dot{X} > \dot{Y}$.

3. Complex Pythagorean Fuzzy Prioritized Aczel–Alsina Operators

To compute the theory of PAOs based on AA t-norm and t-conorm for managing the theory of C-PF information, such as C-PFPAAA, C-PFPAAOA, C-PFPAAG, and C-PFPAAOG operators. Here, we also examined the fundamental properties of the derived theory.

Definition 4: The mathematical form of the C-PFPAAA operator is particularized by:

$$C - PFPAAA(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_k) = \frac{T_1}{\sum_{j=1}^{n} T_j}(\dot{X}_1) \oplus \dots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j}(\dot{X}_n) = \oplus_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n} T_j}(\dot{X}_j)\right) \quad (8)$$

where $T_1 = 1$ and $T_j = \oplus_{k=1}^{j-1} \acute{S}(\dot{X}_k), k = 2,3, \dots, n$.

Theorem 1: Here, we stated that the finalized value of Eq. (8) is again a C-PF information, such as:

$$C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=$$
$$\left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu_{\dot{X}_{RP_j}}^2\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}},\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu_{\dot{X}_{IP_j}}^2\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}}\right),\\ \left(e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}},e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right) \quad (9)$$

Proof of Theorem 1 is provided Appendix-1.

Preposition 1 (*idempotency*): When $\dot{X}_j=\dot{X}=\left(\left(\mu_{\dot{X}_{RP}},\mu_{\dot{X}_{IP}}\right),\left(\nu_{\dot{X}_{RP}},\nu_{\dot{X}_{IP}}\right)\right),j=1,2,\dots,n$, then

$$C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)=\dot{X} \quad (10)$$

Preposition 2 (*monotonicity*): When $\dot{X}_j\le\dot{X}_j^{**},j=1,2,\dots,n$, then

$$C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)\le C-PFPAAA(\dot{X}_1^{**},\dot{X}_2^{**},\dots,\dot{X}_n^{**}) \quad (11)$$

Preposition 3 (*boundedness*): When $\dot{X}_j^{-}=\left(\left(\min_j\mu_{\dot{X}_{RP_j}},\min_j\mu_{\dot{X}_{IP_j}}\right),\left(\max_j\nu_{\dot{X}_{RP_j}},\max_j\nu_{\dot{X}_{IP_j}}\right)\right)$, and $\dot{X}_j^{+}=\left(\left(\max_j\mu_{\dot{X}_{RP_j}},\max_j\mu_{\dot{X}_{IP_j}}\right),\left(\min_j\nu_{\dot{X}_{RP_j}},\min_j\nu_{\dot{X}_{IP_j}}\right)\right),j=1,2,\dots,n$, then

$$\dot{X}_j^{-}\le C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)\le\dot{X}_j^{+} \quad (12)$$

Definition 5: The mathematical form of the C-PFPAAOA operator is particularized by:

$$C-PFPAAOA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=\frac{T_1}{\sum_{j=1}^{n}T_j}\left(\dot{X}_{O(1)}\right)\oplus\dots\oplus\frac{T_n}{\sum_{j=1}^{n}T_j}\left(\dot{X}_{O(n)}\right)=\oplus_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\left(\dot{X}_{O(j)}\right)\right) \quad (13)$$

where $O(j)\le O(j-1),T_1=1$ and $T_j=\oplus_{k=1}^{j-1}\acute{S}(\dot{X}_k),k=2,3,\dots,n$.

Theorem 2: Here, we stated that the finalized value of Eq. (13) is again a C-PF information, such as

$$C-PFPAAOA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=\left(\left(\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_{O(j)}}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}},\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_{O(j)}}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}}\right),\left(e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_{O(j)}}}\right)\right)^{\daleth}\right)^{1/\daleth}},e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_{O(j)}}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\right) \tag{14}$$

Preposition 4 (*idempotency*): When $\dot{X}_j=\dot{X}=\left(\left(\mu_{\dot{X}_{RP}},\mu_{\dot{X}_{IP}}\right),\left(\nu_{\dot{X}_{RP}},\nu_{\dot{X}_{IP}}\right)\right), j=1,2,\dots,n$, then

$$C-PFPAAOA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)=\dot{X} \tag{15}$$

Preposition 5 (*monotonicity*): When $\dot{X}_j\le\dot{X}_j^{**}, j=1,2,\dots,n$, then

$$C-PFPAAOA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)\le C-PFPAAOA(\dot{X}_1^{**},\dot{X}_2^{**},\dots,\dot{X}_n^{**}) \tag{16}$$

Preposition 6 (*boundedness*): When $\dot{X}_j^{-}=\left(\left(\min_j\mu_{\dot{X}_{RP_j}},\min_j\mu_{\dot{X}_{IP_j}}\right),\left(\max_j\nu_{\dot{X}_{RP_j}},\max_j\nu_{\dot{X}_{IP_j}}\right)\right)$, and $\dot{X}_j^{+}=\left(\left(\max_j\mu_{\dot{X}_{RP_j}},\max_j\mu_{\dot{X}_{IP_j}}\right),\left(\min_j\nu_{\dot{X}_{RP_j}},\min_j\nu_{\dot{X}_{IP_j}}\right)\right), j=1,2,\dots,n$, then

$$\dot{X}_j^{-}\le C-PFPAAOA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_n)\le\dot{X}_j^{+} \tag{17}$$

Definition 6: The mathematical form of the C-PFPAAG operator is particularized by:

$$C-PFPAAG(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=(\dot{X}_1)^{\frac{T_1}{\sum_{j=1}^{n}T_j}}\oplus(\dot{X}_2)^{\frac{T_2}{\sum_{j=1}^{n}T_j}}\oplus\dots\oplus(\dot{X}_n)^{\frac{T_n}{\sum_{j=1}^{n}T_j}}=\oplus_{j=1}^{n}\left((\dot{X}_j)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right) \tag{18}$$

where $T_1=1$ and $T_j=\oplus_{k=1}^{j-1}\acute{S}(\dot{X}_k), k=2,3,\dots,n$.

Theorem 3: Here, we stated that the finalized value of Eq. (18) is again a C-PF information

$$C-PFPAAG(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=\left(\left(e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\mu_{\dot{X}_{RP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}},e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\mu_{\dot{X}_{IP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right),\left(\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\nu^2_{\dot{X}_{RP_j}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}},\sqrt{1-e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\nu^2_{\dot{X}_{IP_j}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}}\right)\right) \tag{19}$$

Preposition 7 (*idempotency*): When $\dot{X}_j = \dot{X} = \left(\left(\mu_{\dot{X}_{RP}}, \mu_{\dot{X}_{IP}}\right), \left(\nu_{\dot{X}_{RP}}, \nu_{\dot{X}_{IP}}\right)\right), j = 1,2,\dots,n$, then

$$C - PFPAAG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) = \dot{X} \tag{20}$$

Preposition 8 (*monotonicity*): When $\dot{X}_j \leq \dot{X}_j^{**}, j = 1,2,\dots,n$, then

$$C - PFPAAG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) \leq C - PFPAAG(\dot{X}_1^{**}, \dot{X}_2^{**}, \dots, \dot{X}_n^{**}) \tag{21}$$

Preposition 9 (*boundedness*): When $\dot{X}_j^- = \left(\left(\min_j \mu_{\dot{X}_{RP_j}}, \min_j \mu_{\dot{X}_{IP_j}}\right), \left(\max_j \nu_{\dot{X}_{RP_j}}, \max_j \nu_{\dot{X}_{IP_j}}\right)\right)$, and $\dot{X}_j^+ = \left(\left(\max_j \mu_{\dot{X}_{RP_j}}, \max_j \mu_{\dot{X}_{IP_j}}\right), \left(\min_j \nu_{\dot{X}_{RP_j}}, \min_j \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2,\dots,n$, then

$$\dot{X}_j^- \leq C - PFPAAG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) \leq \dot{X}_j^+ \tag{22}$$

Definition 7: The mathematical form of the C-PFPAAOG operator is particularized by

$$C - PFPAAOG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_k) = \left(\dot{X}_{O(1)}\right)^{\frac{T_1}{\sum_{j=1}^{n} T_j}} \oplus \dots \oplus \left(\dot{X}_{O(n)}\right)^{\frac{T_n}{\sum_{j=1}^{n} T_j}} = \oplus_{j=1}^{n}\left(\left(\dot{X}_{O(j)}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}}\right) \tag{23}$$

where $O(j) \leq O(j-1), T_1 = 1$ and $T_j = \oplus_{k=1}^{j-1} \acute{S}(\dot{X}_k), k = 2,3,\dots,n$.

Theorem 4: Here, we stated that the finalized value of Eq. (23) is again a C-PF information as

$$C - PFPAAOG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_k) = \left(\begin{array}{c}\left(e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(\mu_{\dot{X}_{RP_{O(j)}}}\right)\right)^{ך}\right)^{1/ך}}, e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(\mu_{\dot{X}_{IP_{O(j)}}}\right)\right)^{ך}\right)^{1/ך}}\right), \\ \left(\sqrt{1 - e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(1-\nu^2_{\dot{X}_{RP_{O(j)}}}\right)\right)^{ך}\right)^{\frac{1}{ך}}}}, \sqrt{1 - e^{-\left(\sum_{j=1}^{n}\left(\frac{T_j}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(1-\nu^2_{\dot{X}_{IP_{O(j)}}}\right)\right)^{ך}\right)^{\frac{1}{ך}}}}\right)\end{array}\right) \tag{24}$$

Preposition 10 (*idempotency*): When $\dot{X}_j = \dot{X} = \left(\left(\mu_{\dot{X}_{RP}}, \mu_{\dot{X}_{IP}}\right), \left(\nu_{\dot{X}_{RP}}, \nu_{\dot{X}_{IP}}\right)\right), j = 1,2,\dots,n$, then

$$C - PFPAAOG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) = \dot{X} \tag{25}$$

Preposition 11 (*monotonicity*): When $\dot{X}_j \leq \dot{X}_j^{**}, j = 1,2,\dots,n$, then

$$C - PFPAAOG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) \leq C - PFPAAOG(\dot{X}_1^{**}, \dot{X}_2^{**}, \dots, \dot{X}_n^{**}) \tag{26}$$

Preposition 12 (*boundedness*): When $\dot{X}_j^- = \left(\left(\min_j \mu_{\dot{X}_{RP_j}}, \min_j \mu_{\dot{X}_{IP_j}}\right), \left(\max_j \nu_{\dot{X}_{RP_j}}, \max_j \nu_{\dot{X}_{IP_j}}\right)\right)$, and $\dot{X}_j^+ = \left(\left(\max_j \mu_{\dot{X}_{RP_j}}, \max_j \mu_{\dot{X}_{IP_j}}\right), \left(\min_j \nu_{\dot{X}_{RP_j}}, \min_j \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2, \dots, n$, then

$$\dot{X}_j^- \leq C - PFPAAOG(\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n) \leq \dot{X}_j^+ \quad (27)$$

4. Multiple-Attribute Decision-Making Method-based on the Derived Theory

For evaluating various real-life problems, we illustrated an MADM technique based on the evaluated theory to describe the worth of the derived theory. Then, we demonstrated some numerical examples to show the comparison between the derived work and prevailing approaches.

Assume a family of alternatives $\dot{X}_{AL_1}, \dot{X}_{AL_2}, \dots, \dot{X}_{AL_m}$ and the family of attributes $\dot{X}_1, \dot{X}_2, \dots, \dot{X}_n$. Further, we construct the information matrix while each value of each attribute in every alternative is in the form of C-PFVs, where $\mu_{\dot{X}_{RP}}(\dot{w}), \mu_{\dot{X}_{IP}}(\dot{w}), \nu_{\dot{X}_{RP}}(\dot{w}), \nu_{\dot{X}_{IP}}(\dot{w}) \in [0,1]$ with $0 \leq \mu^2_{\dot{X}_{RP}}(\dot{w}) + \nu^2_{\dot{X}_{RP}}(\dot{w}) \leq 1$ and $0 \leq \mu^2_{\dot{X}_{IP}}(\dot{w}) + \nu^2_{\dot{X}_{IP}}(\dot{w}) \leq 1$. Moreover, $\hat{r}_{\dot{X}}(\dot{w}) = \left(\hat{r}_{\dot{X}_{PR}}(\dot{w}), \hat{r}_{\dot{X}_{IP}}(\dot{w})\right) = \left(\sqrt{1 - \left(\mu^2_{\dot{X}_{RP}}(\dot{w}) + \nu^2_{\dot{X}_{RP}}(\dot{w})\right)}, \sqrt{1 - \left(\mu^2_{\dot{X}_{IP}}(\dot{w}) + \nu^2_{\dot{X}_{IP}}(\dot{w})\right)}\right)$ represented the neutral grade where the simple form of PF value (PFV) is stated by $\dot{X} = \left(\left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right), \left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right)\right), j = 1,2,..,n$. To find the finest preference from the collection of preferences, we use the following algorithm.

Step 1: We arrange the information matrix by including their values in the form of C-PFVs. If the matrix covered the cost types of data, then by using the below theory, we needed to normalize it, such as:

$$M = \begin{cases} \left(\left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right), \left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right)\right) & \text{for benefit} \\ \left(\left(\nu_{\dot{X}_{RP_j}}, \nu_{\dot{X}_{IP_j}}\right), \left(\mu_{\dot{X}_{RP_j}}, \mu_{\dot{X}_{IP_j}}\right)\right) & \text{for cost} \end{cases} \quad (28)$$

Further, if the matrix covered the benefit type of data, then we would not need to normalize it.

Step 2: Here, we aggregate the information matrix by using the theory of the C-PFPAAA operator and the C-PFPAAG operator.

Step 3: Discover the score values or accuracy values of each aggregated value.

Step 4: Find the ranking results and derive the finest optimal from the collection of preferences.

5. Illustrative Example: A Case for a Software Company

A software company wants to appoint an expert for the post of manager. For this, the owner of the company calls five different candidates $\dot{X}_{AL_1}, \dot{X}_{AL_2}, \dot{X}_{AL_3}, \dot{X}_{AL_4}$, and $\dot{X}_{AL_5}$, which are stated as a collection of five alternatives. Further, to choose the most suitable and perfect candidates for the post of manager, the owner of the company looks for the following features such as: $\dot{X}_1, \dot{X}_2, \dot{X}_3$, and $\dot{X}_4$, which represented the collection of four attributes. Then, to find the finest preference from the collection of preferences, we use the proposed MADM algorithm, such as:

Step 1: We arrange the information matrix in the form of Table 1 by including their values in the form of C-PFVs. If the matrix covers the cost types of data, then we need to normalize it. Further, if the matrix covers the benefit type of data, then we do not need to normalize it. The information in Table 1 does not need to be normalized.

Table 1
Original C-PF information matrix

	$\dot{X}_1$	$\dot{X}_2$	$\dot{X}_3$	$\dot{X}_4$
$\dot{X}_{AL_1}$	((0.8,0.7),(0.1,0.2))	((0.81,0.71),(0.11,0.21))	((0.82,0.72),(0.12,0.22))	((0.83,0.73),(0.13,0.23))
$\dot{X}_{AL_2}$	((0.2,0.5),(0.2,0.4))	((0.21,0.51),(0.21,0.41))	((0.22,0.52),(0.22,0.42))	((0.23,0.53),(0.23,0.43))
$\dot{X}_{AL_3}$	((0.7,0.8),(0.2,0.3))	((0.71,0.81),(0.21,0.31))	((0.72,0.82),(0.22,0.32))	((0.73,0.83),(0.23,0.33))
$\dot{X}_{AL_4}$	((0.4,0.5),(0.3,0.4))	((0.41,0.51),(0.31,0.41))	((0.42,0.52),(0.32,0.42))	((0.43,0.53),(0.33,0.43))
$\dot{X}_{AL_5}$	((0.5,0.8),(0.2,0.3))	((0.51,0.81),(0.21,0.31))	((0.52,0.82),(0.22,0.32))	((0.53,0.83),(0.23,0.33))

Step 2: Here, we aggregate the information matrix by using the theory of the C-PFPAAA operator and the C-PFPAAG operator (Table 2).

Table 2
Aggregated information matrix

	$\mathbf{C-PFPAAA}$	$\mathbf{C-PFPAAG}$
$\dot{X}_{AL_1}$	((0.8186,0.7185),(0.1170,0.2174))	((0.8174,0.7175),(0.1196,0.2189))
$\dot{X}_{AL_2}$	((0.2058,0.5184),(0.2174,0.4175))	((0.2046,0.5176),(0.2189,0.4185))
$\dot{X}_{AL_3}$	((0.7182,0.8186),(0.2174,0.3175))	((0.7172,0.8174),(0.2189,0.3186))
$\dot{X}_{AL_4}$	((0.4083,0.5184),(0.3175,0.4175))	((0.4073,0.5176),(0.3186,0.4185))
$\dot{X}_{AL_5}$	((0.5169,0.8186),(0.2174,0.3175))	((0.5159,0.8174),(0.2189,0.3186))

Step 3: Discover the score values or accuracy values of each aggregated value (Table 3).

Table 3
Score information matrix

	$\mathbf{C-PFPAAA}$	$\mathbf{C-PFPAAG}$
$\dot{X}_{AL_1}$	0.5627	0.5604
$\dot{X}_{AL_2}$	0.0447	0.433
$\dot{X}_{AL_3}$	0.5189	0.5165
$\dot{X}_{AL_4}$	0.0801	0.0785
$\dot{X}_{AL_5}$	0.3946	0.3924

Step 4: Find the ranking results and derive the finest from the collection of preferences (Table 4).

Table 4
Ranking information matrix

Methods	Rankings
$\mathbf{C-PFPAAA}$	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_4} \leq \dot{X}_{AL_2}$
$\mathbf{C-PFPAAG}$	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_2} \leq \dot{X}_{AL_4}$

According to the theory of the C-PFPAAA operator and the C-PFPAAG operator, we get the best candidate for the software company as $\dot{X}_{AL_1}$.

Moreover, we finalized the supremacy and effectiveness of the derived theory with the help of some comparative analysis under the consideration of the above case study example. In Table 5, according to the theory of C-PFPAAA and C-PFPAAG, as well as Jin *et al.* [27] and Akram *et al.* [26], we noticed that the best candidate for the software company is $\dot{X}_{AL_1}$.

Table 5
Representation of the comparative analysis

Methods	Score values	Rankings
Akram *et al.* [26]	0.5622,0.0471,0.5185,0.0829,0.3945	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_4} \leq \dot{X}_{AL_2}$
Average AO [27]	0.5552,0.0467,0.5129,0.083,0.3904	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_4} \leq \dot{X}_{AL_2}$
Geometric AO [27]	0.5536,0.0457,0.5112,0.0818,0.3889	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_2} \leq \dot{X}_{AL_4}$
C-PFPAAA	0.5627,0.0447,0.5189,0.0801,0.3946	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_4} \leq \dot{X}_{AL_2}$
C-PFPAAG	0.5604,0.433,0.5165,0.0785,0.3924	$\dot{X}_{AL_1} \leq \dot{X}_{AL_3} \leq \dot{X}_{AL_5} \leq \dot{X}_{AL_2} \leq \dot{X}_{AL_4}$

6. Conclusion

C-PF information is an extended form of fuzzy set theory, which covers the grade of membership and the grade of non-membership with a valuable characteristic that is the sum of the squares of the duplet (for real and imaginary parts) should be contained in the unit interval. In this analysis, we provided the following advances:

i. the theory of the C-PFPAAA operator and the C-PFPAAOA operator;
ii. the theory of the C-PFPAAG operator and the C-PFPAAOG operator;
iii. examined the idempotency, monotonicity, and boundedness properties of those AOs;
iv. illustrated a MADM technique to describe the reliability and effectiveness of the AOs;
v. demonstrated some numerical examples to state the capability of the new AOs.

In the future, we will extend the theory of prioritized AA AOs for C-PF information into complex T-spherical fuzzy sets and try to employ it in the field of pattern recognition and clustering analysis.

Appendix-1: Proof of Theorem 1

Here, with the help of mathematical induction, we prove that the information in Eq. (8). For this, we assume $n = 2$, then we have

$$\left(\frac{T_1}{\sum_{j=1}^{n} T_j}\right)\dot{X}_1 = \left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\left(\left(\frac{T_1}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(1-\mu_{\dot{X}_{RP_1}}^2\right)\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1-e^{-\left(\left(\left(\frac{T_1}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(1-\mu_{\dot{X}_{IP_1}}^2\right)\right)\right)^{\daleth}\right)^{1/\daleth}}}\right), \\ \left(e^{-\left(\left(\left(\frac{T_1}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_1}}\right)\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(\left(\frac{T_1}{\sum_{j=1}^{n} T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_1}}\right)\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right)$$

$$\left(\frac{T_2}{\sum_{j=1}^{n} T_j}\right)\dot{X}_2 = \left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1-e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}\right), \\ \left(e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right)$$

Thus

$$C-PFPAAA(\dot{X}_1, \dot{X}_2) = \frac{T_1}{\sum_{j=1}^{n}T_j}(\dot{X}_1) \oplus \frac{T_2}{\sum_{j=1}^{n}T_j}(\dot{X}_2) =$$

$$\left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\left(\frac{T_1}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_1}}\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1-e^{-\left(\left(\frac{T_1}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_1}}\right)\right)^{\daleth}\right)^{1/\daleth}}}\right), \\ \left(e^{-\left(\left(\frac{T_1}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_1}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(\frac{T_1}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_1}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right) \oplus$$

$$\left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}, \sqrt{1-e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}}\right), \\ \left(e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(\frac{T_2}{\sum_{j=1}^{n}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_2}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left(\sqrt{1-e^{-\left(\sum_{j=1}^{2}\left(\frac{T_j}{\sum_{j=1}^{2}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_j}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}}, \sqrt{1-e^{-\left(\sum_{j=1}^{2}\left(\frac{T_j}{\sum_{j=1}^{2}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_j}}\right)\right)^{\daleth}\right)^{\frac{1}{\daleth}}}}\right), \\ \left(e^{-\left(\sum_{j=1}^{2}\left(\frac{T_j}{\sum_{j=1}^{2}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\sum_{j=1}^{2}\left(\frac{T_j}{\sum_{j=1}^{2}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_j}}\right)\right)^{\daleth}\right)^{1/\daleth}}\right)\end{array}\right)$$

For $n=2$, we got the best result. Further, for $n=k$, we also considered that our result is correct. Then

$$C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_k)=$$

$$\left(\left(\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}},\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}}\right),\left(e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth},e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth}\right)\right)$$

Then, for $n=k+1$, we derive our required theme, such as

$$C-PFPAAA(\dot{X}_1,\dot{X}_2,\dots,\dot{X}_{k+1})=\frac{T_1}{\sum_{j=1}^{n}T_j}(\dot{X}_1)\oplus\frac{T_2}{\sum_{j=1}^{n}T_j}(\dot{X}_2)\oplus\dots\oplus\frac{T_k}{\sum_{j=1}^{k}T_j}(\dot{X}_k)\oplus\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}(\dot{X}_{k+1})$$

$$=\oplus_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{n}T_j}(\dot{X}_j)\right)\oplus\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}(\dot{X}_{k+1})$$

$$=\left(\left(\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}},\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}}\right),\left(e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth},e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth}\right)\right)\oplus\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}(\dot{X}_{k+1})$$

$$\left(\left(\sqrt{1-e^{-\left(\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_{k+1}}}\right)\right)\right)^{\daleth}}}^{1/\daleth},\sqrt{1-e^{-\left(\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_{k+1}}}\right)\right)\right)^{\daleth}}}^{1/\daleth}\right),\left(e^{-\left(\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_{k+1}}}\right)\right)\right)^{\daleth}}{}^{1/\daleth},e^{-\left(\left(\frac{T_{k+1}}{\sum_{j=1}^{k+1}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_{k+1}}}\right)\right)\right)^{\daleth}}{}^{1/\daleth}\right)\right)$$

$$=\left(\left(\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}},\sqrt{1-e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(1-\mu^2_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}}\right),\left(e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{RP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth},e^{-\left(\sum_{j=1}^{k}\left(\frac{T_j}{\sum_{j=1}^{k}T_j}\right)\left(-ln\left(\nu_{\dot{X}_{IP_j}}\right)\right)\right)^{\daleth}}{}^{1/\daleth}\right)\right)\oplus$$

$$= \left(\begin{array}{c} \left(\sqrt{1 - e^{-\left(\sum_{j=1}^{k+1}\left(\frac{T_j}{\sum_{j=1}^{k+1} T_j}\right)\left(-ln\left(1-\mu^2_{\check{X}_{RP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}}, \sqrt{1 - e^{-\left(\sum_{j=1}^{k+1}\left(\frac{T_j}{\sum_{j=1}^{k+1} T_j}\right)\left(-ln\left(1-\mu^2_{\check{X}_{IP_j}}\right)\right)\right)^{\daleth}}}^{\frac{1}{\daleth}} \right), \\ \left(e^{-\left(\left(\sum_{j=1}^{k+1}\left(\frac{T_j}{\sum_{j=1}^{k+1} T_j}\right)\left(-ln\left(\nu_{\check{X}_{RP_j}}\right)\right)\right)^{\daleth}\right)^{1/\daleth}}, e^{-\left(\left(\sum_{j=1}^{k+1}\left(\frac{T_j}{\sum_{j=1}^{k+1} T_j}\right)\left(-ln\left(\nu_{\check{X}_{IP_j}}\right)\right)\right)^{\daleth}\right)^{1/\daleth}} \right) \end{array} \right)$$

Hence, our required result is proven.

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Conflicts of Interest

The author declares no conflicts of interest.

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