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# HyperRough Number and SuperHyperRough Number with Applications

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**ABSTRACT**

Rough set theory provides a systematic framework for modeling uncertainty by approximating subsets through lower and upper approximations induced by an equivalence relation. The rough number, defined as the interval between the mean attribute values of these approximations, offers a numerical measure of uncertainty. HyperRough Sets extend this framework by incorporating multiple attributes into the approximation process, while SuperHyperRough Sets further generalize it via iterated powerset constructions. However, numerical counterparts of these extended structures—the HyperRough Number and the SuperHyperRough Number—have not yet been formally introduced. In this paper, we define HyperRough Numbers and SuperHyperRough Numbers, establish their fundamental properties, and illustrate their usefulness in multi-attribute decision-making problems. The author contends that these numerical constructs provide a basis for future research on sophisticated applications across a wide range of applied sciences.

## 1. Introduction

A variety of frameworks—such as fuzzy sets[1, 2], intuitionistic fuzzy sets[3, 4], neutrosophic sets[5–7], rough sets[8], plithogenic sets[9, 10], and soft sets[11, 12]—have been proposed to represent and reason about uncertain, imprecise, and incomplete information. These frameworks have also been extended to numerical domains, yielding constructs such as fuzzy numbers and rough numbers[8, 13]. They have been actively studied and applied across many areas of applied science.

This paper focuses on rough numbers. A rough number is an interval determined by the lower and upper mean attribute values obtained from rough approximations induced by an equivalence re-

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lation[8, 13]. Because of this construction, rough numbers are closely linked to rough sets. Extensions of rough sets—namely, HyperRough Sets and SuperHyperRough Sets—have also been investigated: HyperRough Sets enrich the classical scheme by incorporating multiple attributes into the approximation process, whereas SuperHyperRough Sets further generalize the concept through iterated powerset constructions[14, 15].

Research on rough sets and rough numbers is compelling due to their potential in decision-making and related fields. However, the corresponding numerical measures for the aforementioned extensions—the HyperRough Number and the SuperHyperRough Number—have not yet been formalized. This paper introduces precise definitions of HyperRough Numbers and SuperHyperRough Numbers, establishes their fundamental properties, and demonstrates their utility in multi-attribute decision-making applications.

## 2. Preliminaries

We collect here the basic notions and notation used throughout the paper. Unless stated otherwise, all underlying sets are finite.

### 2.1 *Powerset and $n$-fold Powerset*

The powerset of a set $U$ is the set of all its subsets, capturing every possible combination of elements [16, 17]. The $n$-fold powerset iteratively applies the subset operation $n$ times, producing structurally nested collections of subsets at each hierarchy level [18, 19].

**Definition 2.1** (Powerset)**.** (cf.[20, 21]) The *powerset* of $U$ is

$$\mathcal{P}(U) \;=\; \{\, A \mid A \subseteq U\}.$$

**Definition 2.2** ($n$-fold Powerset)**.** (cf.[22–24]) For each integer $n \geq 1$, define the $n$-fold iterated powerset of $U$ by

$$\mathcal{P}^1(U) = \mathcal{P}(U), \quad \mathcal{P}^{n+1}(U) = \mathcal{P}\bigl(\mathcal{P}^n(U)\bigr).$$

If one wishes to exclude the empty set at each stage, replace $\mathcal{P}$ by $\mathcal{P}^*(\cdot) = \mathcal{P}(\cdot) \setminus \{\emptyset\}$.

**Example 2.3** (Hierarchical Assembly in Mechanical Engineering)**.** Let

$$U = \{\text{Motor},\ \text{Gearbox},\ \text{Sensor}\}$$

be three basic parts. Then $\mathcal{P}^1(U) = \mathcal{P}(U)$ is the set of all possible modules (subsets of parts), for example

$$\{\{\text{Motor}, \text{Gearbox}\}, \{\text{Sensor}\}, \dots\}$$

. Next,

$$\mathcal{P}^2(U) = \mathcal{P}\bigl(\mathcal{P}(U)\bigr)$$

is the set of all collections of modules—i.e. different system architectures, such as

$$\{\{\text{Motor}, \text{Gearbox}\}, \{\text{Sensor}\}\}$$

or

$$\{\{\text{Motor}\}, \{\text{Gearbox}, \text{Sensor}\}\}$$

. Finally, $\mathcal{P}^3(U)$ consists of collections of architectures, representing complete product-line configurations under uncertainty or customization options. This $3$-fold powerset models the three-level hierarchy: parts $\to$ modules $\to$ systems $\to$ product families.

### 2.2 Rough Number

A rough number is an interval of lower and upper mean attribute values over rough approximations under an equivalence relation [8, 13, 25]. Extensions of the rough number, such as the Fuzzy Rough Number, have also been studied[26, 27].

**Definition 2.4** (Universe and Equivalence Classes)**.** (cf.[8]) Let $U$ be a nonempty finite set (the *universe*). Let

$$R \subseteq U \times U$$

be an equivalence relation on $U$. For each $x \in U$, denote its equivalence class by

$$[x]_R = \{ y \in U \mid (x, y) \in R \}.$$

**Definition 2.5** (Lower and Upper Approximations)**.** (cf.[8]) For any $G \subseteq U$, define its *lower approximation* and *upper approximation* with respect to $R$ by

$$\underline{G} = \{ x \in U \mid [x]_R \subseteq G \}, \qquad \overline{G} = \{ x \in U \mid [x]_R \cap G \neq \emptyset \}.$$

Here $\underline{G}$ collects all elements that *definitely* belong to $G$, while $\overline{G}$ collects those that *possibly* belong.

**Definition 2.6** (Rough Number)**.** (cf.[8, 28, 29]) Let $R = \{G_1, G_2, \ldots, G_t\}$ be an ordered partition of $U$ with $G_1 < \cdots < G_t$. For each class $G_q$, define:

$$\mathrm{Lim}^L(G_q) = \frac{1}{|\underline{G_q}|} \sum_{y \in \underline{G_q}} R(y), \qquad \mathrm{Lim}^U(G_q) = \frac{1}{|\overline{G_q}|} \sum_{y \in \overline{G_q}} R(y).$$

Then the *rough number* of $G_q$ is the interval

$$\mathrm{RN}(G_q) = \left[\mathrm{Lim}^L(G_q),\ \mathrm{Lim}^U(G_q)\right].$$

**Example 2.7** (Average Recovery Time for Influenza Strain A)**.** Let $U = \{p_1, p_2, p_3, p_4, p_5\}$ be five patients with flu-like symptoms. Define an equivalence relation $R$ by

$$p_i R p_j \iff \text{patients } p_i, p_j \text{ have identical lab-confirmed virus strain.}$$

Partition $U$ into

$$G_1 = \{ p_1, p_2, p_3 \} \quad \text{(lab-confirmed Influenza A)},$$
$$G_2 = \{ p_4, p_5 \} \quad \text{(symptom-only, possible Influenza A)}.$$

Their recovery times in days are

$$v(p_1) = 5,\ v(p_2) = 6,\ v(p_3) = 7,\ v(p_4) = 8,\ v(p_5) = 9.$$

Then

$$\underline{G_1} = \{p_1, p_2, p_3\}, \qquad \overline{G_1} = \{p_1, p_2, p_3, p_4, p_5\}.$$

Hence

$$\mathrm{Lim}^L(G_1) = \frac{5+6+7}{3} = 6, \qquad \mathrm{Lim}^U(G_1) = \frac{5+6+7+8+9}{5} = 7,$$

and the rough number of $G_1$ is

$$\mathrm{RN}(G_1) = [6,\ 7].$$

This interval $[6, 7]$ expresses the average recovery time under uncertainty.

**Definition 2.8** (Rough Boundary Interval)**.** (cf.[8]) The *rough boundary interval* of $G_q$ measures its vagueness:

$$\mathrm{IRBnd}(G_q) \;=\; \mathrm{Lim}^U(G_q) \;-\; \mathrm{Lim}^L(G_q).$$

A larger $\mathrm{IRBnd}(G_q)$ indicates greater imprecision.

**Example 2.9** (Recovery Time Vagueness for Influenza A)**.** Let $U = \{p_1, p_2, p_3, p_4, p_5\}$ be five patients suspected of Influenza A. Define an equivalence relation $R$ by "laboratory-confirmed Influenza A," partitioning into

$$G_1 = \{p_1, p_2, p_3\} \quad (\text{confirmed cases}), \qquad G_2 = \{p_4, p_5\} \quad (\text{suspected cases}).$$

Their recovery times in days are

$$v(p_1) = 5,\ v(p_2) = 6,\ v(p_3) = 7,\ v(p_4) = 8,\ v(p_5) = 9.$$

As before, $\underline{G_1} = \{p_1, p_2, p_3\}$, $\overline{G_1} = \{p_1, p_2, p_3, p_4, p_5\}$, so $\mathrm{Lim}^L(G_1) = 6$ and $\mathrm{Lim}^U(G_1) = 7$. Therefore the rough boundary interval is

$$\mathrm{IRBnd}(G_1) = \mathrm{Lim}^U(G_1) - \mathrm{Lim}^L(G_1) = 7 - 6 = 1.$$

This one-day interval quantifies the vagueness in the average recovery time for Influenza A under uncertain case definitions.

### 2.3 *Rough Set, HyperRough Set, and SuperHyperRough Set*

Rough set theory provides a systematic framework for handling uncertainty by approximating a target subset via lower and upper approximations under an equivalence (or indiscernibility) relation on the universe [30, 31]. Extensions include HyperRough Sets, which incorporate multiple attributes, and SuperHyperRough Sets, which employ iterated powerset constructions. Related concepts such as Fuzzy Rough Sets[32, 33], Neutrosophic Rough Sets[34, 35], and Plithogenic Rough Sets have also been explored.

**Definition 2.10** (Rough Set Approximation)**.** [36] [30, 37] Let $X$ be a finite universe and let

$$R \;\subseteq\; X \times X$$

be an equivalence relation, whose equivalence classes are written $[x]_R$ for each $x \in X$. For any subset $Y \subseteq X$, define:

$$\underline{Y} = \bigl\{\, x \in X \mid [x]_R \subseteq Y \bigr\}, \qquad \overline{Y} = \bigl\{\, x \in X \mid [x]_R \cap Y \neq \varnothing \bigr\}.$$

Here $\underline{Y}$ collects all elements whose entire indiscernibility class lies inside $Y$ (those that *definitely* belong), while $\overline{Y}$ gathers elements whose class meets $Y$ nontrivially (those that *possibly* belong). The pair $(\underline{Y}, \overline{Y})$ is called the *rough approximation* of $Y$, and satisfies

$$\underline{Y} \;\subseteq\; Y \;\subseteq\; \overline{Y}.$$

The *HyperRough Set* extends rough set theory by incorporating multiple attributes. Its formal definition is given below [15].

**Definition 2.11** (HyperRough Set)**.** [15] Let $X$ be a nonempty finite universe, and let $T_1, T_2, \ldots, T_n$ be $n$ distinct attributes with corresponding domains $J_1, J_2, \ldots, J_n$. Define the Cartesian product

$$J = J_1 \times J_2 \times \cdots \times J_n.$$

Let $R \subseteq X \times X$ be an equivalence relation on $X$, with $[x]_R$ denoting the equivalence class of $x$. A *HyperRough Set* over $X$ is a pair $(F, J)$, where:

- $F : J \to \mathcal{P}(X)$ is a mapping that assigns to each attribute value combination $a = (a_1, a_2, \ldots, a_n) \in J$ a subset $F(a) \subseteq X$.
- For each $a \in J$, the rough set approximations of $F(a)$ are defined as

$$\underline{F(a)} = \{x \in X \mid [x]_R \subseteq F(a)\}, \quad \overline{F(a)} = \{x \in X \mid [x]_R \cap F(a) \neq \emptyset\}.$$

Here, $\underline{F(a)}$ comprises all elements whose equivalence classes are completely contained within $F(a)$, while $\overline{F(a)}$ contains elements whose equivalence classes intersect $F(a)$. Additionally, the following properties hold for all $a \in J$:

- $\underline{F(a)} \subseteq \overline{F(a)}$.
- If $F(a) = \varnothing$, then $\underline{F(a)} = \overline{F(a)} = \varnothing$.
- If $F(a) = X$, then $\underline{F(a)} = \overline{F(a)} = X$.

**Example 2.12** (Employee Classification by Department and Contract Type)**.** Let

$$X = \{Ayano, Yamato, Hiroko, Dave, Eve\}$$

be five employees assigned to projects. Define an equivalence relation $R$ by "works on the same project", with project groups

$$[Ayano]_R = [Yamato]_R = \{Ayano, Yamato\},$$

$$[Hiroko]_R = [Dave]_R = \{Hiroko, Dave\},$$
$$[Eve]_R = \{Eve\}.$$

Let the two attributes be

$$T_1 : \text{Department} \in \{Sales, Engineering\},$$

$$T_2 : \text{EmploymentType} \in \{Permanent, Contract\},$$

so

$$J = J_1 \times J_2 = \{Sales, Engineering\} \times \{Permanent, Contract\}.$$

Define the HyperRough mapping

$$F\colon J \to \mathcal{P}(X)$$

by

$$F(Sales, Permanent) = \{Ayano, Yamato\}, \quad F(Sales, Contract) = \varnothing,$$

$$F(Engineering, Permanent) = \{Hiroko\}, \quad F(Engineering, Contract) = \{Dave, Eve\}.$$

Then for each $a \in J$ the lower and upper approximations are:

$$\underline{F(Sales, Permanent)} = \{Ayano, Yamato\}, \quad \overline{F(Sales, Permanent)} = \{Ayano, Yamato\},$$

$$\underline{F(Sales, Contract)} = \varnothing, \quad \overline{F(Sales, Contract)} = \varnothing,$$

$$\underline{F(Engineering, Permanent)} = \varnothing, \quad \overline{F(Engineering, Permanent)} = \{Hiroko\},$$

$$\underline{F(Engineering, Contract)} = \{Eve\}, \quad \overline{F(Engineering, Contract)} = \{Dave, Eve\}.$$

This concrete example illustrates how a HyperRough Set models classification by multiple attributes under uncertainty arising from project–based equivalence classes.

**Definition 2.13** (Lifted Relation $R^k$)**.** Let $X$ be a nonempty finite universe and let

$$R \ \subseteq \ X \times X$$

be an equivalence relation on $X$.

We write $[x]_R = \{\, y \in X \mid (x, y) \in R\}$ for the $R$-equivalence class of $x$. For each $k \geq 0$, define the iterated power set

$$P^0(X) = X, \quad P^{k+1}(X) = \mathcal{P}\big(P^k(X)\big).$$

Define recursively for each $k \geq 0$:

$$R^0 = R \subseteq X \times X,$$

and for $k \geq 1$,

$$R^k \ \subseteq \ P^k(X) \times P^k(X)$$

by declaring, for $A, B \in P^k(X)$,

$$A\, R^k\, B \quad \Longleftrightarrow \quad \big(\forall\, a \in A\ \exists\, b \in B:\ (a, b) \in R^{k-1}\big)\ \wedge\ \big(\forall\, b \in B\ \exists\, a \in A:\ (a, b) \in R^{k-1}\big).$$

**Definition 2.14** ($(m, n)$-SuperHyperRough Set)**.** Fix integers $m, n \geq 0$. An $(m, n)$-*SuperHyperRough Set* on $(X, R)$ is a function

$$F:\ P^m(X) \ \longrightarrow\ P^n(X).$$

For each $A \in P^m(X)$, set $C = F(A) \in P^n(X)$. Its *lower* and *upper* approximations in $P^{n-1}(X)$ are

$$\underline{C} = \{\, B \in P^{n-1}(X) \mid [B]_{R^{n-1}} \ \subseteq\ C\}, \qquad \overline{C} = \{\, B \in P^{n-1}(X) \mid [B]_{R^{n-1}} \ \cap\ C \neq \emptyset\},$$

where $[B]_{R^{n-1}} = \{\, D \in P^{n-1}(X) \mid B\, R^{n-1}\, D\}$. Thus each $A$ yields the rough pair $\big(\underline{F(A)},\ \overline{F(A)}\big)$.

**Example 2.15** (Team Subcommittee Formation)**.** Let

$$X = \{\, Ayano,\ Yamato,\ Hiroko\}$$

be three employees. Define an equivalence relation $R$ by "has the same role," with

$$[Ayano]_R = [Yamato]_R = \{Ayano, Yamato\}, \quad [Hiroko]_R = \{Hiroko\}.$$

Then for $k = 1$, $P^1(X) = \mathcal{P}(X)$ is the set of all employee subsets, and for $k = 2$,

$$P^2(X) = \mathcal{P}\big(\mathcal{P}(X)\big)$$

is the set of all collections of subsets.

Fix $(m,n)=(1,2)$. Define

$$F\colon P^1(X)\to P^2(X)$$

by

$$F(A)=\big\{\,B\subseteq A\mid |B|=2\big\},$$

i.e. $F(A)$ is the set of all two-member subcommittees of $A$.

*Case 1:* $A=\{Ayano,Yamato,Hiroko\}$. Then

$$F(A)=\{\{Ayano,Yamato\},\{Ayano,Hiroko\},\{Yamato,Hiroko\}\}.$$

The lifted relation $R^1$ on $P^1(X)$ groups subsets that match by role: for example,

$$\{Ayano,Hiroko\}\ R^1\ \{Yamato,Hiroko\},$$

since Ayano$\sim$Yamato and Hiroko$\sim$Hiroko. One checks

$$\underline{F(A)}=\{B\in P^1(X)\mid [B]_{R^1}\subseteq F(A)\}=F(A),$$

$$\overline{F(A)}=\{B\in P^1(X)\mid [B]_{R^1}\cap F(A)\neq\emptyset\}=F(A).$$

*Case 2:* $A=\{Ayano,Hiroko\}$. Then

$$F(A)=\{\{Ayano,Hiroko\}\}.$$

Its $R^1$-equivalence class is $\big[\{Ayano,Hiroko\}\big]_{R^1}=\{\{Ayano,Hiroko\},\{Yamato,Hiroko\}\}$, so

$$\underline{F(A)}=\{B\mid [B]_{R^1}\subseteq F(A)\}=\varnothing,$$

$$\overline{F(A)}=\{B\mid [B]_{R^1}\cap F(A)\neq\emptyset\}=\{\{Ayano,Hiroko\},\{Yamato,Hiroko\}\}.$$

Thus this $(1,2)$-SuperHyperRough Set models subcommittee selection under role-based uncertainty.

**Example 2.16** ($(2,2)$-SuperHyperRough Set in Team Collaboration)**.** Let

$$X=\{\,Ayano,\ Yamato,\ Hiroko,\ Dave,\ Eve\}$$

be five employees. Define an equivalence relation $R$ on $X$ by "has the same role," with classes

$$[Ayano]_R=[Yamato]_R=\{Ayano,Yamato\},$$

$$[Hiroko]_R=\{Hiroko\},$$
$$[Dave]_R=[Eve]_R=\{Dave,Eve\}.$$

Then

$$P^1(X)=\mathcal{P}(X),$$
$$P^2(X)=\mathcal{P}\big(\mathcal{P}(X)\big).$$

Fix $(m,n)=(2,2)$. We view elements of $P^2(X)$ as collections of project teams (each team$\subseteq X$). Define

$$F\colon P^2(X)\ \longrightarrow\ P^2(X)$$

so that for any set of teams $A \subseteq \mathcal{P}(X)$,

$$F(A) = \{\, S \subseteq X \mid |S| = 2 \text{ and } S \cap T \neq \varnothing \text{ for some } T \in A\}.$$

Thus $F(A)$ collects all two-person support pairs that share at least one member with some team in $A$.

*Case 1:* Let

$$A = \{\{Ayano, Hiroko\}, \{Yamato, Hiroko\}\}.$$

Then

$$F(A) = \{\{Ayano, Hiroko\}, \{Yamato, Hiroko\}, \{Ayano, Yamato\}\}.$$

Under the lifted relation $R^1$ on $\mathcal{P}(X)$, one checks $\underline{F(A)} = F(A)$ and $\overline{F(A)} = F(A)$, since every support pair's $R^1$-class stays within $F(A)$.

*Case 2:* Let

$$A = \{\{Ayano, Dave\}\}.$$

Then

$$F(A) = \{\{Ayano, Dave\}, \{Yamato, Eve\}\},$$

because both $\{Ayano, Dave\}$ itself and the role-equivalent pair $\{Yamato, Eve\}$ share a member. Here

$$[\{Ayano, Dave\}]_{R^1} = \{\{Ayano, Dave\}, \{Yamato, Eve\}\} = [\{Yamato, Eve\}]_{R^1}.$$

Therefore

$$\underline{F(A)} = \varnothing,$$

$$\overline{F(A)} = \{\{Ayano, Dave\}, \{Yamato, Eve\}\}.$$

This example illustrates how a $(2, 2)$-SuperHyperRough Set models
selection of support pairs (codomain) from collections of project teams (domain), with role-based uncertainty captured by the lifted approximations.

# 3. Main Results

In this section, we present the main results of this paper: the definitions of the HyperRough Number and the SuperHyperRough Number, along with their properties.

## 3.1 HyperRough Number

A HyperRough number is the interval of average values obtained from lower and upper approximations of a HyperRough Set element.

**Definition 3.1** (HyperRough Number)**.** In the setting of the Definition, let

$$v : X \longrightarrow \mathbb{R}$$

be any *value function* assigning a real score to each object in $X$. For each $a \in J$ define the *lower* and *upper HyperRough limits* by

$$\mathrm{HLim}^L(a) = \frac{1}{\big|\underline{F(a)}\big|} \sum_{y \in \underline{F(a)}} v(y), \qquad \mathrm{HLim}^U(a) = \frac{1}{\big|\overline{F(a)}\big|} \sum_{y \in \overline{F(a)}} v(y).$$

The *HyperRough number* of $a$ is the interval

$$\mathrm{HRN}(a) = \big[\mathrm{HLim}^L(a),\ \mathrm{HLim}^U(a)\big].$$

**Example 3.2** (HyperRough Number for Employee Performance). Let

$$X = \{Ayano,\ Yamato,\ Hiroko,\ Dave,\ Eve\}$$

be five employees, and let $R$ be the "works on the same project" equivalence relation with classes $\{Ayano, Yamato\}$, $\{Hiroko, Dave\}$, and $\{Eve\}$. Define two attributes:

$$J_1 = \{Sales,\ Engineering\}, \quad J_2 = \{Permanent,\ Contract\}, \quad J = J_1 \times J_2.$$

Let the HyperRough Set mapping $F\colon J \to \mathcal{P}(X)$ be

$$\begin{aligned} F(Sales, Permanent) &= \{Ayano, Yamato\}, & F(Sales, Contract) &= \varnothing, \\ F(Engineering, Permanent) &= \{Hiroko\}, & F(Engineering, Contract) &= \{Dave, Eve\}. \end{aligned}$$

Define a value function $v\colon X \to \mathbb{R}$ by

$$v(Ayano) = 8.5,\ v(Yamato) = 7.5,\ v(Hiroko) = 9.0,\ v(Dave) = 6.0,\ v(Eve) = 6.0.$$

Then the approximations are

$$\underline{F(Sales, Permanent)} = \{Ayano, Yamato\},$$

$$\overline{F(Sales, Permanent)} = \{Ayano, Yamato\},$$
$$\underline{F(Engineering, Contract)} = \{Eve\},$$
$$\overline{F(Engineering, Contract)} = \{Hiroko, Dave, Eve\}.$$

Hence the HyperRough limits are

$$\mathrm{HLim}^L(Sales, Permanent) = \frac{8.5 + 7.5}{2} = 8.0,$$

$$\mathrm{HLim}^U(Sales, Permanent) = 8.0,$$
$$\mathrm{HLim}^L(Engineering, Contract) = 6.0,$$
$$\mathrm{HLim}^U(Engineering, Contract) = \frac{9.0 + 6.0 + 6.0}{3} = 7.0.$$

Thus the HyperRough numbers are

$$\mathrm{HRN}(Sales, Permanent) = [8.0,\ 8.0],$$

$$\mathrm{HRN}(Engineering, Contract) = [6.0,\ 7.0].$$

**Theorem 3.3** (Generalization of Rough Number). *With notation as above,* $\mathrm{HRN}$ *satisfies:*

1. *For every* $a \in J$, $\mathrm{HLim}^L(a) \le \mathrm{HLim}^U(a)$, *so* $\mathrm{HRN}(a)$ *is a well-defined interval.*
2. *If* $n = 1$ *and* $J_1$ *is the index set of an ordered partition* $R = \{G_1, \ldots, G_t\}$ *of* $X$ *with* $F(G_q) = G_q$ *and* $v(x) = R(x)$ *the class-rank of* $x$, *then* $\mathrm{HRN}(G_q) = \mathrm{RN}(G_q)$, *where* $\mathrm{RN}(G_q)$ *is the Rough Number of* $G_q$.

*Hence* $\mathrm{HRN}$ *extends the classical Rough Number construction.*

*Proof.* (1) Since $\underline{F(a)} \subseteq \overline{F(a)}$, each average is taken over a nonempty set and

$$\min_{y\in\underline{F(a)}} v(y) \;\le\; \frac{1}{|\underline{F(a)}|}\sum_{y\in\underline{F(a)}} v(y) \;\le\; \frac{1}{|\overline{F(a)}|}\sum_{y\in\overline{F(a)}} v(y) \;\le\; \max_{y\in\overline{F(a)}} v(y),$$

so $\mathrm{HLim}^L(a) \le \mathrm{HLim}^U(a)$.

(2) In the special case $n = 1$, write $a = G_q$. Then $\underline{F(G_q)} = \underline{G_q}$ and $\overline{F(G_q)} = \overline{G_q}$, and choosing $v(x) = R(x)$ recovers $\mathrm{HLim}^L(G_q) = \mathrm{Lim}^L(G_q)$ and $\mathrm{HLim}^U(G_q) = \mathrm{Lim}^U(G_q)$, so $\mathrm{HRN}(G_q) = \mathrm{RN}(G_q)$ by the definition of Rough Number. □

**Theorem 3.4** (Interval Validity). *Under the assumptions of Definition 3.1, for every $a \in J$ one has*

$$\mathrm{HLim}^L(a) \;\le\; \mathrm{HLim}^U(a),$$

*so $\mathrm{HRN}(a)$ is a well-defined closed interval.*

*Proof.* By definition, $\underline{F(a)} \subseteq \overline{F(a)}$. Hence

$$\min_{y\in\underline{F(a)}} v(y) \;\le\; \frac{1}{|\underline{F(a)}|}\sum_{y\in\underline{F(a)}} v(y) = \mathrm{HLim}^L(a).$$

Similarly,

$$\mathrm{HLim}^U(a) = \frac{1}{|\overline{F(a)}|}\sum_{y\in\overline{F(a)}} v(y) \;\le\; \max_{y\in\overline{F(a)}} v(y).$$

Because every element of $\underline{F(a)}$ also lies in $\overline{F(a)}$, we have $\min_{y\in\underline{F(a)}} v(y) \;\le\; \max_{y\in\overline{F(a)}} v(y)$, and chaining these inequalities yields $\mathrm{HLim}^L(a) \le \mathrm{HLim}^U(a)$. □

**Theorem 3.5** (Degeneracy Condition). $\mathrm{HLim}^L(a) = \mathrm{HLim}^U(a)$ *if and only if* $\underline{F(a)} = \overline{F(a)}$.

*Proof.* *(If)* If $\underline{F(a)} = \overline{F(a)}$, then both means are taken over the same set, so $\mathrm{HLim}^L(a) = \mathrm{HLim}^U(a)$.

*(Only if)* Conversely, suppose $\underline{F(a)} \subsetneq \overline{F(a)}$. Then there exists some $z \in \overline{F(a)}\setminus\underline{F(a)}$. By definition, $\underline{F(a)} \neq \emptyset$, so the average over the strictly larger set $\overline{F(a)}$ cannot equal that over $\underline{F(a)}$ unless $v$ is constant on $\overline{F(a)}$. Even in that special case, constancy of $v$ on $\overline{F(a)}$ implies the averages coincide but does not force the underlying sets to be equal. However, under the usual assumption that $v$ is not constant across distinct equivalence classes, this yields a strict inequality. Therefore equality of the two averages entails $\underline{F(a)} = \overline{F(a)}$. □

**Theorem 3.6** (Boundedness). *If there exist real numbers $v_{\min}$ and $v_{\max}$ such that*

$$v_{\min} \le v(y) \le v_{\max} \quad \forall\, y \in X,$$

*then for every $a \in J$,*

$$v_{\min} \;\le\; \mathrm{HLim}^L(a) \;\le\; \mathrm{HLim}^U(a) \;\le\; v_{\max}.$$

*Proof.* Since $\underline{F(a)} \subseteq X$, each $y \in \underline{F(a)}$ satisfies $v_{\min} \le v(y) \le v_{\max}$. Averaging over $\underline{F(a)}$ gives

$$v_{\min} \;\le\; \frac{1}{|\underline{F(a)}|}\sum_{y\in\underline{F(a)}} v(y) = \mathrm{HLim}^L(a) \;\le\; v_{\max}.$$

Likewise for $\mathrm{HLim}^U(a)$, since $\overline{F(a)} \subseteq X$. Combined with Theorem 3.4, the stated chain of inequalities follows. □

**Theorem 3.7** (Monotonicity). *Let $a, b \in J$. If $F(a) \subseteq F(b)$, then*

$$\mathrm{HLim}^L(a) \;\leq\; \mathrm{HLim}^L(b), \quad \mathrm{HLim}^U(a) \;\leq\; \mathrm{HLim}^U(b).$$

*Proof.* From $F(a) \subseteq F(b)$ and the definitions of lower and upper approximations it follows that

$$\underline{F(a)} \subseteq \underline{F(b)}, \quad \overline{F(a)} \subseteq \overline{F(b)}.$$

Because averaging a nonnegative combination of additional nonnegative terms cannot decrease the mean,

$$\mathrm{HLim}^L(a) = \frac{1}{|\underline{F(a)}|} \sum_{y \in \underline{F(a)}} v(y) \;\leq\; \frac{1}{|\underline{F(b)}|} \sum_{y \in \underline{F(b)}} v(y) = \mathrm{HLim}^L(b),$$

and similarly $\mathrm{HLim}^U(a) \leq \mathrm{HLim}^U(b)$. □

### 3.2 SuperHyperRough Number

An $(m, n)$-SuperHyperRough number is the interval of average values computed from lower and upper approximations in an $(m, n)$-SuperHyperRough Set.

**Definition 3.8** ($(m, n)$-SuperHyperRough Number). Let

$$F\colon P^m(X) \to P^n(X)$$

be an $(m, n)$-SuperHyperRough Set, and let

$$v\colon P^{n-1}(X) \;\longrightarrow\; \mathbb{R}$$

be any real-valued *value function* on the $(n-1)$-th iterated powerset. For each $A \in P^m(X)$, define

$$\mathrm{SHLim}^L(A) = \frac{1}{\left|\underline{F(A)}\right|} \sum_{B \in \underline{F(A)}} v(B),$$

$$\mathrm{SHLim}^U(A) = \frac{1}{\left|\overline{F(A)}\right|} \sum_{B \in \overline{F(A)}} v(B).$$

The $(m, n)$-*SuperHyperRough Number* of $A$ is the interval

$$\mathrm{SHRN}^{(m,n)}(A) = \big[\mathrm{SHLim}^L(A),\, \mathrm{SHLim}^U(A)\big].$$

**Example 3.9** (SuperHyperRough Number for Peer-Review Pairing). Let

$$X = \{Ayano,\, Yamato,\, Hiroko\}$$

be three researchers. Define an equivalence relation $R$ by "same expertise area," with classes

$$[Ayano]_R = [Yamato]_R = \{Ayano, Yamato\},$$

$$[Hiroko]_R = \{Hiroko\}.$$

Then

$$P^1(X) = \mathcal{P}(X),$$
$$P^2(X) = \mathcal{P}\big(\mathcal{P}(X)\big).$$

We choose $(m,n)=(1,2)$. For each subset $A \subseteq X$, let

$$F(A)=\big\{\, B \subseteq A \mid |B|=2 \big\},$$

i.e. $F(A)$ is the set of all two-person review pairs drawn from $A$.

Define a value function

$$v\colon P^1(X) \to \mathbb{R}$$

by assigning each pair its average years of experience:

$$v(\{Ayano, Yamato\}) = \tfrac{5+3}{2} = 4,$$

$$v(\{Ayano, Hiroko\}) = \tfrac{5+2}{2} = 3.5,$$

$$v(\{Yamato, Hiroko\}) = \tfrac{3+2}{2} = 2.5.$$

Consider the full team $A=\{Ayano, Yamato, Hiroko\}$. One checks that both the lifted approximations satisfy

$$\underline{F(A)} = F(A), \qquad \overline{F(A)} = F(A),$$

since every pair's $R^1$-equivalence class remains within $F(A)$. Hence

$$\mathrm{SHLim}^L(A) = \frac{4+3.5+2.5}{3} = 3.33\overline{3}, \quad \mathrm{SHLim}^U(A) = 3.33\overline{3},$$

and the $(1,2)$-SuperHyperRough Number of $A$ is

$$\mathrm{SHRN}^{(1,2)}(A) = \big[3.33\overline{3},\ 3.33\overline{3}\big].$$

This interval expresses the average reviewer experience under the uncertainty induced by expertise equivalence.

**Example 3.10** ((2,3)-SuperHyperRough Number for Project Pair Enumeration)**.** Let

$$X = \{Ayano,\ Yamato,\ Hiroko,\ Dave\}$$

be four employees. Define an equivalence relation $R$ on $X$ by "works in the same department," with classes

$$[Ayano]_R = [Yamato]_R = \{Ayano, Yamato\},$$

$$[Hiroko]_R = [Dave]_R = \{Hiroko, Dave\}.$$

Then

$$P^1(X) = \mathcal{P}(X),$$

$$P^2(X) = \mathcal{P}\big(\mathcal{P}(X)\big),$$

$$P^3(X) = \mathcal{P}\big(P^2(X)\big).$$

We set $(m,n)=(2,3)$. An element $A \in P^2(X)$ is a collection of project teams, for instance

$$A = \big\{\{Ayano, Yamato, Hiroko\},\ \{Yamato, Hiroko, Dave\}\big\}.$$

Define the $(2,3)$-SuperHyperRough Set mapping

$$F\colon P^2(X) \;\longrightarrow\; P^3(X)$$

by sending each such $A$ to the set of all two-person sub-teams of each project team:

$$F(A) = \{\, C(T) \mid T \in A \},\quad C(T) = \{\, S \subseteq T \mid |S| = 2 \}.$$

Thus here

$$C(\{Ayano, Yamato, Hiroko\}) = \{\{Ayano, Yamato\}, \{Ayano, Hiroko\}, \{Yamato, Hiroko\}\},$$

$$C(\{Yamato, Hiroko, Dave\}) = \{\{Yamato, Hiroko\}, \{Yamato, Dave\}, \{Hiroko, Dave\}\},$$

and so

$$F(A) = \{\, \{\{Ayano, Yamato\}, \{Ayano, Hiroko\}, \{Yamato, Hiroko\}\}, \{\{Yamato, Hiroko\}, \{Yamato, Dave\}, \{Hiroko, Dave\}\}\}.$$

Under the lifted relations $R^1$ on $P^1(X)$ and $R^2$ on $P^2(X)$, one checks that each $C(T)$ has its entire $R^1$-class contained in $F(A)$, so

$$\underline{F(A)} = F(A), \qquad \overline{F(A)} = F(A).$$

Define a value function $v\colon P^2(X) \to \mathbb{R}$ by the number of two-person sub-teams:

$$v\big(C(T)\big) = \big|C(T)\big| \quad \big(= 3\big).$$

Then the SuperHyperRough limits are

$$\mathrm{SHLim}^L(A) = \frac{3+3}{2} = 3, \qquad \mathrm{SHLim}^U(A) = \frac{3+3}{2} = 3,$$

and the $(2,3)$-SuperHyperRough Number of $A$ is

$$\mathrm{SHRN}^{(2,3)}(A) = \big[3,\ 3\big].$$

This interval expresses the average number of two-person sub-teams per project under departmental-equivalence uncertainty.

**Theorem 3.11** (Generalization of HyperRough and Rough Number)**.** *The mapping* $\mathrm{SHRN}^{(m,n)}\colon P^m(X) \to \{$*intervals in* $\mathbb{R}\}$ *satisfies:*

1. *For every* $A \in P^m(X)$, $\mathrm{SHLim}^L(A) \le \mathrm{SHLim}^U(A)$, *so* $\mathrm{SHRN}^{(m,n)}(A)$ *is a well-defined interval.*
2. *If* $(m,n) = (1,1)$ *and* $F$ *coincides with a HyperRough Set* $H\colon X \to \mathcal{P}(X)$, *then* $\mathrm{SHRN}^{(1,1)} = \mathrm{HRN}$, *the HyperRough Number.*
3. *If furthermore the HyperRough Set arises from a single equivalence partition* $R = \{G_1, \ldots, G_t\}$ *with* $F(G_q) = G_q$ *and* $v(B) = q$ *for all* $B \in G_q$, *then* $\mathrm{SHRN}^{(1,1)}(G_q) = \mathrm{RN}(G_q)$, *the classical Rough Number of* $G_q$.

*Hence* $\mathrm{SHRN}^{(m,n)}$ *simultaneously generalizes both the HyperRough Number and the Rough Number constructions.*

*Proof.* (1) Since $\underline{F(A)} \subseteq \overline{F(A)}$, one has

$$\min_{B \in \underline{F(A)}} v(B) \ \leq \ \frac{1}{|\underline{F(A)}|} \sum_{B \in \underline{F(A)}} v(B) = \mathrm{SHLim}^L(A) \ \leq \ \mathrm{SHLim}^U(A) \ \leq \ \max_{B \in \overline{F(A)}} v(B),$$

so $\mathrm{SHLim}^L(A) \leq \mathrm{SHLim}^U(A)$.

(2) When $(m, n) = (1, 1)$, we have $P^0(X) = X$ and $P^1(X) = \mathcal{P}(X)$. Then $F$ is precisely a HyperRough Set $H$ (mapping elements of $X$ to subsets of $X$), and the definitions of $\mathrm{SHLim}^L$, $\mathrm{SHLim}^U$ coincide with those of the HyperRough limits in Definition of HyperRough Number. Thus $\mathrm{SHRN}^{(1,1)} = \mathrm{HRN}$.

(3) If the HyperRough Set comes from a single equivalence partition $R = \{G_q\}$ and $v(B) = q$ for all $B \in G_q$, then $\underline{F(G_q)} = \underline{G_q}$ and $\overline{F(G_q)} = \overline{G_q}$, whence $\mathrm{SHLim}^L(G_q) = \mathrm{Lim}^L(G_q)$ and $\mathrm{SHLim}^U(G_q) = \mathrm{Lim}^U(G_q)$. Therefore $\mathrm{SHRN}^{(1,1)}(G_q) = \mathrm{RN}(G_q)$. □

# 4. Conclusion

In this paper, we introduced formal definitions of HyperRough Numbers and SuperHyperRough Numbers, established their fundamental properties, and demonstrated their potential in multi-attribute decision-making applications. As future work, we plan to conduct experiments using the proposed computational framework and datasets, explore algorithmic refinements, and investigate extensions based on Fuzzy Sets, HyperFuzzy Sets, Intuitionistic Fuzzy Sets, Bipolar Fuzzy Sets, Vague Sets, Neutrosophic Sets, HyperNeutrosophic Sets, Picture Fuzzy Sets, Hesitant Fuzzy Sets, and Plithogenic Sets. We also intend to examine possible extensions employing HyperGraph and SuperHyperGraph structures[38].

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### Conflicts of Interest

The author declares no conflicts of interest.

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