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An Introduction of the Mellin Transformation under Interval Uncertainty

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ABSTRACT

Mellin transformation has potential applications in signal processing, quantum mechanics, and numerous domains of knowledge. Data collection and its interpretations must go through impreciseness in almost every real field of decision-making or modelling. In this paper, we contribute an alternative approach for the Mellin transformation under imprecision. The interval-valued Mellin transformation is introduced in this paper. We discuss the linearity, scaling, and sifting properties of the interval-valued transformation. Also, the Mellin transformation of the generalized Hukuhara derivative of the interval-valued function is addressed. One possible application of interval-valued Mellin transformation to solve an uncertain Euler differential equation is manifested, and several more consequent applications are hinted.

1. Introduction

Integral transforms are widely recognized as effective mathematical tools for solving complicated problems arising in science and engineering. Recent developments have introduced new transform frameworks and generalized formulations that enhance the analytical treatment of differential equations and mathematical models [1, 2]. Moreover, integral transform techniques have been successfully applied in mathematical physics and engineering, where they provide efficient approaches for analyzing complex systems and boundary value problems [3, 4]. Several researchers have also introduced new integral transforms to enhance the solution methodology for differential and integral equations.

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For example, the Sumudu transform [5], N-transform [6], Elzaki transform [7], and Upadhyaya transform [8] were proposed as alternatives to classical transforms with improved computational flexibility in specific applications. Furthermore, generalized and fractional transform frameworks have been developed to address complex mathematical models involving nonlocal operators and fractional dynamics, including the general integral transform of Jafari [9], fractional Fourier transform formulations [10, 11], and generalized fractional Fourier transforms [12, 13]. The Mellin transform is an important integral transform that has extensive applications in mathematics and engineering, especially in the areas of asymptotic analysis, number theory, and complex analysis. Its earliest known use appears in a memoir by Riemann, where the transform was applied in the investigation of the well-known Zeta function. Further details regarding this work, along with extensions introduced by M. Cahen, are discussed in [14]. Although the transformation originated earlier, it was the Finnish mathematician R. H. Mellin (1854–1933) who systematically developed both the transform and its inverse. In the study of special functions, Mellin employed this technique to solve hypergeometric differential equations and to derive asymptotic expansions. His work strongly emphasized analytic function theory and made significant use of Cauchy's theorem together with the residue method. A detailed account of Mellin's life and contributions is available in [15]. It serves as a powerful tool for solving differential and integral equations and is closely related to other transforms such as the Fourier and Laplace transforms. The concept of the interval Mellin transform extends the traditional Mellin transform by incorporating interval analysis, which deals with intervals rather than precise numerical values. This extension is particularly useful in handling uncertainties and imprecise data, providing a framework for robust mathematical and engineering analyses.

The non-deterministic nature of the dynamical systems given by differential and integral equation necessitated the uncertain counterpart of the proposed theory. In this paper, we are aiming to establish a theory of interval valued Mellin transform. In this connection, we surveyed the literature on the Mellin transform under uncertainty. We found a very few papers on fuzzy Mellin transformations. Sun & Yang [16] introduced the fuzzy Mellin transformation and establihsed its associations with two sided Laplace transformation. They also addressed the applicability of their proposed theory for solving the fuzzy differential equations. Azhar & Iqbal [17] incorporated the fractional sense in fuzzy Mellin transformation. Furthermore, they used it to solve fuzzy fractional differential equations. Singh *et al.* [18] discussed the fuzzy Mellin transformation in the context of modified Hukuhara derivative. Apart from this direction, uncertain Mellin transformation were used in crystallization of impreciseness monitored multi criteria group decision making. In this context, Jafariani & Tabatabaee [19] discussed joint Fourier-Mellin transformation with fuzzy clustering. Type 2 interval valued fuzzy set has a significant role in decision making problems under vagueness. Gong *et al.* [20] used Mellin transformation for interpretation Type 2 interval fuzzy numbers as Mellin transformation of Type 1 fuzzy numbers. Chen & Huan [21] discussed trade-off between mean and spread using defuzzifcation technique associated with uncertain environment. Several worthy investigations [22, 23] were credited following the mentioned approach.

The study of Interval analysis is a mathematical framework for dealing with ranges (intervals) of values instead of precise numbers. It provides methods to handle uncertainties, rounding errors, and imprecise measurements systematically. In interval analysis, numbers are represented as intervals $[a, b]$, where all values within the interval are considered possible. Operations are defined to propagate these intervals through mathematical computations, ensuring that the true value lies within the resulting interval. Interval analysis introduced by [24]. Numerous works were done on the optimization oriented manifestation of the interval analysis. Now, we surveyed the literature of interval analysis with the keywords integral transformation under interval uncertainty for estimating progress in this direction. It is to be noted that only a few studies on interval valued integral transformations were published till date. In this context, [25] addressed fast Fourier transformation under interval uncertainty. Also, Das *et al.* [26] contributed a novel study of interval Laplace transformation and its possible implications on the lot size models. Up to the authors' knowledge, there is no single article in literature regarding the interval valued Mellin transformation. In this paper, the interval valued Mellin transformation is introduced. Various impacting characteristics like the linearity, scale and sifting of the Mellin transformation has been discussed in details. The result of the interval valued Mellin transformation for the generalized Hukuhara derivative are categorized. With the introduction of of interval valued Mellin transformation, this paper may be regarded as a pioneer in this direction of future research.

The remaining of this article is organized as follows: Section 2 describes the mathematical prerequisite for the proposed theory. The main contribution of this article is given in Section 3, Section 4 and Section 5. In Section 3, we introduce the interval valued Mellin transformation. Different properties, like linearity, scaling and sifting are addressed under the newly proposed definition of Mellin transformation in Section 4. In Section 5, we discuss the Mellin transformation of the generalized Hukuhara derivatives of interval valued function. Here, the possible applications of the interval valued Mellin transformation are also hinted. Ending remarks on the overall findings and contributions in this article are given in Section 6.

2. Preliminaries

In this present section, we discuss the mathematical prerequisite regarding the interval numbers and interval valued calculus. Here is the beginning with the arithmetic properties of the interval numbers. Let κ_c denote the family of all bounded closed intervals in $\mathbb{R}$; i.e., let $\kappa_c = \{X = [\chi_L, \chi_U] : \chi_L, \chi_U \in \mathbb{R}$ and $\chi_L \leq \chi_U\}$. Some interval arithmetic rules on κ_c are defined below:

(i) $X + Y = [\chi_L + \xi_L, \chi_U + \xi_U]$

(ii) $X - Y = X + (-1)Y = [\chi_L - \xi_L, \chi_U - \xi_U]$

(iii) $\lambda X = \begin{cases} [\lambda\chi_L, \lambda\chi_U],\ \lambda \geq 0 \\ [\lambda\chi_U, \lambda\chi_L],\ \lambda < 0 \end{cases}$

(iv) $XY = [u, v]$, where $u = min\{\chi_L\xi_L, \chi_L\xi_U, \chi_U\xi_L, \chi_U\xi_U\}$ and $v = max\{\chi_L\xi_L, \chi_L\xi_U, \chi_U\xi_L, \chi_U\xi_U\}$

(v) $\frac{1}{X} = [\frac{1}{\chi_U}, \frac{1}{\chi_L}]$, provided $0 \notin X$

(vi) $X \leq Y \Leftrightarrow \chi_L \leq \xi_L, \chi_U \leq \xi_U$

Definition 1. The generalized Hukuhara difference of two intervals X and Y, denoted by $X \ominus_{gH} Y$, is defined as:

$$X \ominus_{gH} Y = Z = \begin{cases} (a)\ \ X = Y + Z, \quad or \\ (b)\ \ Y = X + (-1)Z \end{cases} \tag{1}$$

This difference has many interesting properties, for example $X \ominus_{gH} Y = 0 = [0, 0]$. Also, the gH-difference of two intervals $X = [\chi_L, \chi_U]$ and $Y = [\xi_L, \xi_U]$ always exists and it is equal to $X \ominus_{gH} Y = [min\{\chi_L - \xi_L\}, max\{\chi_U - \xi_U\}]$. Henceforth, S will denote an open subset of $\mathbb{R}$. Let $G : S \longrightarrow \kappa_c$ be an interval-valued function with $G(x) = [g_L(x), g_U(x)]$, where $g_L(x) \leq g_U(x), \forall x \in S$. The functions

g_L and g_U are called the lower and the upper endpoint functions of G, respectively.

Definition 2. Let $x_0 \in S$ and let $G : S \longrightarrow \kappa_c$ be an interval-valued function, then the generalized Hukuhara derivative of G at x_0 is defined as:

$$G'(x_0) = \lim_{h \to 0} \frac{G(x_0 + h) \ominus_{gH} G(x_0)}{h} \tag{2}$$

If $G'(x_0) \in \kappa_c$ satisfying Eq. (2) exists, we say that G is generalized Hukuhara differentiable at x_0. One-sided derivatives can also be evaluated. The right-hand derivative of G at x_0 is given by:

$$G'(x_0) = \lim_{h \to 0^+} \frac{G(x_0 + h) \ominus_{gH} G(x_0)}{h} \tag{3}$$

while the left-hand derivative can be defined as:

$$G'(x_0) = \lim_{h \to 0^-} \frac{G(x_0 + h) \ominus_{gH} G(x_0)}{h} \tag{4}$$

It is important to note that the gH-derivative exists at x_0 if and only if both the left-hand and right-hand derivatives at x_0 exist and are equal.

Definition 3. Let $[\chi_L, \chi_U] : [0, \infty) \to \kappa_c$ be defined by $\chi_L(x) = min\{M(x), N(x)\}$ and $\chi_U(x) = max\{M(x), N(x)\}, x > 0$; then $x_0 \in [0, \infty)$ is said to be a switching point of $[\chi_L, \chi_U]$ if $M(x) - N(x)$ alters signs at x_0.

Definition 4. Let $G : S \longrightarrow \kappa_c$ be an continuous interval-valued function with $G(x) = [g_L(x), g_U(x)]$ such that $g_L(x) \leq g_U(x)$ for all $x \in S$ and the integral of an interval-valued function G be defined as $\int_a^b G(x)dx = [\int_a^b g_L(x)dx, \int_a^b g_U(x)dx]$. Then:

1. The function $G(x) = \int_a^x g(t)dt$ is differentiable and $G'(x) = g(x)$
2. The function $F(x) = \int_x^b f(t)dt$ is differentiable and $F'(x) = -f(x)$

Corollary 1. If G is gH-differentiable with no switching point in the interval $[a, b]$ then we have $\int_a^b G'(x)dx = f(b) \ominus_{gH} f(a)$.

Definition 5. Let $[\chi_L, \chi_U] : [0, \infty) \to \kappa_c$ be given by $[\chi_L, \chi_U](x) = [\chi_L(x), \chi_U(x)]$ for all $x \in [0, \infty)$. Then, the improper integral $\int_0^\infty [\chi_L, \chi_U](x)dx$ is said to be convergent if and only if both the improper integral $\int_0^\infty \chi_L(x)dx$ and $\int_0^\infty \chi_U(x)dx$ are convergent.

Corollary 2. If the improper integral converges, then:

$$\int_0^\infty [\chi_L, \chi_U](x)dx = \left[\int_0^\infty \chi_L(x)dx, \int_0^\infty \chi_U(x)dx \right] \tag{5}$$

3. Mellin Transforms of Interval-Valued Functions

Definition 6. Let $[\chi_L, \chi_U] : [0, \infty) \to \kappa_c$ be an interval-valued function with $\chi_L(x) \leq \chi_U(x)$ for all $x \geq 0$. For $Re(p) > 0$, the interval Mellin transform of $[\chi_L, \chi_U]$ is defined by:

$$\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = \left[\int_0^\infty x^{p-1} \chi_L(x)\, dx,\ \int_0^\infty x^{p-1} \chi_U(x)\, dx \right] \tag{6}$$

provided both integrals converge.

Definition 7 *(polynomial growth condition)*. An interval-valued function $[\chi_L, \chi_U]$ is said to satisfy the polynomial growth condition if there exist positive constants C_1, C_2, α, and β such that the absolute values of the endpoint functions are bounded by $C_1 x^{-\alpha}$ as $x \to 0^+$ and by $C_2 x^{\beta}$ as $x \to \infty$. That is:

$$|\chi_L(x)|,\ |\chi_U(x)| \leq C_1 x^{-\alpha} \quad (x \to 0^+) \tag{7}$$

and

$$|\chi_L(x)|,\ |\chi_U(x)| \leq C_2 x^{\beta} \quad (x \to \infty) \tag{8}$$

In other words, the endpoint functions exhibit controlled polynomial behavior near the origin and at infinity, ensuring that their growth remains within prescribed bounds.

Theorem 1 (*existence of interval Mellin transform*). The interval Mellin transform $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\}$ exists and defines a valid interval if and only if:

1. The classical Mellin transforms $M\{\chi_L(x); p\}$ and $M\{\chi_U(x); p\}$ exist
2. There exists a vertical strip $0 < Re(p) < \infty$ such that

$$M\{\chi_L(x); p\} \leq M\{\chi_U(x); p\} \quad \text{for all admissible } p \tag{9}$$

In this case $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = \big[M\{\chi_L(x); p\}, M\{\chi_U(x); p\}\big]$.

Proof of Theorem 1: Given in Appendix 1.

Corollary 3. Theorem 1 provides the following results:

1. If the Mellin transform of $[\chi_L, \chi_U]$ exists, then $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = [M\{\chi_L(x); p\}, M\{\chi_U(x); p\}]$
2. Let $\chi_L(x) = \min\{\phi(x), \xi(x)\}$ and $\chi_U(x) = \max\{\phi(x), \xi(x)\}$

If the ordering $\phi(x) \leq \xi(x)$ (or vice versa) holds on $(0, \infty)$, then $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = [M\{\chi_L(x); p\},\ M\{\chi_U(x); p\}]$.

4. Properties of Interval-Valued Mellin Transform

Property 1 (*Linearity property*). Let $[\chi_L, \chi_U]_i : [0, \infty) \to \kappa_c$ be defined by $[\chi_L, \chi_U]_i(x) = [\chi_{iL}(x), \chi_{iU}(x)]$ with $i = 1, 2$ as the two interval valued function whose Mellin transform exists and a and b be two non negative real constants. Then $\mathcal{M}_I\{a[\chi_{1L}(x), \chi_{1U}(x)] + b[\chi_{2L}(x), \chi_{2U}(x)]; p\} = a\mathcal{M}_I\{[\chi_{1L}(x), \chi_{1U}(x)]; p\} + b\mathcal{M}_I\{[\chi_{2L}(x), \chi_{2U}(x)]; p\}$.

Proof of Property 1: Given in Appendix 2.

Corollary 4. Linearity property for the Mellin transform of an interval-valued function also holds when the coefficient is an interval number; i.e., $\mathcal{M}_I\{[a_L, a_U][\chi_{1L}(x), \chi_{1U}(x)] + [b_L, b_U][\chi_{2L}(x), \chi_{2U}(x)]; p\} = [a_L, a_U]\mathcal{M}_I\{[\chi_{1L}(x), \chi_{1U}(x)]; p\} + [b_L, b_U]\mathcal{M}_I\{[\chi_{2L}(x), \chi_{2U}(x)]; p\}$, where $[a_L, a_U], [b_L, b_U] \in \kappa_c$ provided $a_L \geq 0$ and $b_L \geq 0$.

Property 2 (*Scaling property*). Let $[\chi_L, \chi_U] : (0, \infty) \to \kappa_c$ be an interval-valued function whose interval Mellin transform exists. For any real constant $c > 0$, the following relation holds:

$$\mathcal{M}_I\{[\chi_L(cx), \chi_U(cx)]; p\} = \frac{1}{c^p}\mathcal{M}_I\{[\chi_L(x), \chi_U(x)]; p\}, \quad Re(p) > 0 \tag{10}$$

Proof of Property 2: Given in Appendix 3.

Property 3 (*Shifting property*). Let $[\chi_L, \chi_U]$ be an interval-valued function such that both $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\}$ and $\mathcal{M}_I\{[\chi_L, \chi_U](x); p + c\}$ exist and $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = [\tilde{\chi_L}(p), \tilde{\chi_U}(p)]$. Then, for any real number c, it follows that $\mathcal{M}_I\{x^c[\chi_L, \chi_U](x); p\} = [\tilde{\chi_L}(p + c), \tilde{\chi_U}(p + c)]$.

Proof of Property 3: Given in Appendix 4.

Property 4. Let $[\chi_L, \chi_U]$ be an interval-valued function such that $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\}$ and $\mathcal{M}_I\{[\chi_L, \chi_U](x); \frac{p}{c}\}$ both exis and $\mathcal{M}_I\{[\chi_L(x), \chi_U(x)]; p\} = [\tilde{\chi_L}(p), \tilde{\chi_U}(p)]$. Then, for any real number $c > 0$, it follows that $\mathcal{M}_I\{[\chi_L(x^c), \chi_U(x^c)]; p\} = \frac{1}{c}[\tilde{\chi_L}(\frac{p}{c}), \tilde{\chi_U}(\frac{p}{c})]$.

Proof of Property 4: Given in Appendix 5.

5. Mellin Transform gH-Derivatives of Interval-Valued Functions

Theorem 2. Let $[\chi_L, \chi_U] : (0, \infty) \to \kappa_c$ be an interval-valued function that is generalized Hukuhara differentiable. Assume that for $i = L, U$, we have $\lim_{x\to 0^+} x^{p-1}\chi_i(x) = 0,\ \lim_{x\to\infty} x^{p-1}\chi_i(x) = 0,$ and $Re(p) > 1$. Then the interval Mellin transform of the gH-derivative satisfies:

$$\mathcal{M}_I\{[\chi_L, \chi_U]'(x); p\} = \begin{cases} -(p-1)\,\mathcal{M}_I\{[\chi_L, \chi_U](x); p-1\}, & \chi_L'(x) \le \chi_U'(x) \\ -(p-1)\,\mathcal{M}_I\{[\chi_U, \chi_L](x); p-1\}, & \chi_U'(x) \le \chi_L'(x) \end{cases} \tag{11}$$

where the inequalities hold on the corresponding sub-intervals of $(0, \infty)$.

Proof of Theorem 2: Given in Appendix 6.

Example 1. Let $[\chi_L, \chi_U](x) = [e^{-2x}, e^{-x}]$, $x > 0$. For $Re(p) > 0$, we have:

$$\begin{aligned} \mathcal{M}_I\{[e^{-2x}, e^{-x}]; p\} &= \int_0^\infty x^{p-1}[e^{-2x}, e^{-x}]dx \\ &= \left[\int_0^\infty x^{p-1}e^{-2x}dx, \int_0^\infty x^{p-1}e^{-x}dx\right] \\ &= [\frac{1}{2^p}\Gamma(p), \Gamma(p)] \end{aligned} \tag{12}$$

The considered interval-valued function is generalized Hukuhara differentiable of the first kind on the interval $(0, \ln 2)$, while it is generalized Hukuhara differentiable of the second kind on $(\ln 2, \infty)$. Therefore, applying Theorem 2, we obtain the following result:

$$\mathcal{M}_I\{[e^{-2x}, e^{-x}]'(x); p\} = \begin{cases} [-\Gamma(p), -\frac{1}{2^{p-1}}\Gamma(p)], & x \in (0, \ln 2) \\ [-\frac{1}{2^{p-1}}\Gamma(p), -\Gamma(p)], & x \in (\ln 2, \infty) \end{cases}$$

Theorem 3. Let $[\chi_L, \chi_U] : (0, \infty) \to \kappa_c$ be an interval-valued function that is generalized Hukuhara differentiable. Assume that for $i = L, U$, we have $\lim_{x\to 0^+} x^{p-1}\chi_i(x) = 0,\ \lim_{x\to\infty} x^{p-1}\chi_i(x) = 0,$ and $Re(p) > 1$. Then:

$$M_1\{x[\chi_L, \chi_U]'(x); p\} = -pM_1\{[\chi_L, \chi_U](x); p\} \tag{13}$$

and

$$M_1\{x^2[\chi_L, \chi_U]'(x); p\} = p(p+1)M_1\{[\chi_L, \chi_U](x); p\} \tag{14}$$

Proof of Theorem 3: It is straight forward.

Example 2. Consider an interval valued Euler differential equation as follows:

$$\left(x^2\mathcal{D}^2 + 2x\mathcal{D} - 2\right)[\chi_L, \chi_U](x) = [2, 10]\,u(a-x)\,x^{-3}, \quad a > 0 \tag{15}$$

In Eq. (15), $u(a-x)$ is a step function. Then, applying the interval valued Mellin transformation, we have $(p^2 - p - 2)M_1\{[\chi_L, \chi_U](x); p\} = [2, 10]\frac{a^{p-3}}{(p-3)}$.

That is $M_1\{[\chi_L, \chi_U](x); p\} = [2, 10]\frac{a^{p-3}}{(p+1)(p-2)(p-3)} = [2, 10]a^{p-3}[\frac{1}{4(p-3)} - \frac{1}{3(p-2)} - \frac{1}{12(p+1)}]$.

Then, the inverse Mellin transformation provides the solution of the interval valued Euler differential equation as $[\chi_L, \chi_U](x) = [2u(a-x)(\frac{1}{4x^3} - \frac{1}{3x^2} + \frac{x}{12}), 10u(a-x)(\frac{1}{4x^3} - \frac{1}{3x^2} + \frac{x}{12})]$.

6. Conclusions

Mellin transformation is associated with several advanced technological aspects, such as image recognition and scaling moving objects related to the computer science domain. On the other hand, the analytical behavior of the Riemann-Zeta function, Weyl functional integrations, and derivatives can be investigated better with the help of the Mellin transformation. The associated functions in both theoretical and practical aspects may carry imprecision due to measurement, prediction, and decoding intuitions. In such phenomena, the study of the uncertain Mellin transformation is necessitated. This paper contributed a theoretical introduction to the interval-valued Mellin transformation. Furthermore, it also solves an uncertain Euler differential equation as an application of the proposed study, hinting at a path of more such advancements in uncertain ordinary and partial differential equations before being used aptly in computer science and management models.

The present work is limited to the theoretical development of the interval-valued Mellin transform and the investigation of its fundamental properties. Moreover, the applicability of the proposed transform has been illustrated through a single uncertain Euler differential equation. Future research may focus on developing inversion formulas and numerical algorithms, establishing additional operational properties, and extending the proposed framework to fractional differential equations and other uncertain mathematical models arising in science and engineering.

Appendix 1: Proof of Theorem 1

Assume that the interval Mellin transform $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\}$ exists.
By Definition 1, it is given by $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = \left[\int_0^\infty x^{p-1}\chi_L(x)\,dx, \int_0^\infty x^{p-1}\chi_U(x)\,dx\right]$.
Since the transform exists, both improper integrals $\int_0^\infty x^{p-1}\chi_L(x)\,dx$ and $\int_0^\infty x^{p-1}\chi_U(x)\,dx$ converge absolutely. Hence, the classical Mellin transforms $M\{\chi_L(x); p\}$ and $M\{\chi_U(x); p\}$ exist. Moreover, because $x^{p-1} \geq 0$ for $Re(p) > 0$ and $\chi_L(x) \leq \chi_U(x)$ for all $x > 0$, the monotonicity of the integral implies $\int_0^\infty x^{p-1}\chi_L(x)\,dx \leq \int_0^\infty x^{p-1}\chi_U(x)\,dx.$

Therefore, $M\{\chi_L(x); p\} \leq M\{\chi_U(x); p\}$ for all admissible p in the vertical strip $0 < \Re(p) < \infty$. This proves conditions (i) and (ii).

Conversely, assume that the classical Mellin transforms $M\{\chi_L(x); p\}$ and $M\{\chi_U(x); p\}$ exist and satisfy $M\{\chi_L(x); p\} \leq M\{\chi_U(x); p\}$ for all p with $0 < \Re(p) < \infty$.

Then, both integrals $\int_0^\infty x^{p-1}\chi_L(x)\,dx$ and $\int_0^\infty x^{p-1}\chi_U(x)\,dx$ are finite. Since $x^{p-1} \geq 0$ in the admissible strip, the ordering of the endpoints is preserved under integration.

Hence, the expression $\left[\int_0^\infty x^{p-1}\chi_L(x)\,dx, \int_0^\infty x^{p-1}\chi_U(x)\,dx\right]$ defines a valid closed interval in κ_c. Therefore, the interval Mellin transform $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\}$ exists and is given by $\mathcal{M}_I\{[\chi_L, \chi_U](x); p\} = \big[M\{\chi_L(x); p\}, M\{\chi_U(x); p\}\big]$.

Appendix 2: Proof of Property 1

Using the definition of the Mellin transform, we have $\mathcal{M}_I\{a[\chi_{1L}(x), \chi_{1U}(x)] + b[\chi_{2L}(x), \chi_{2U}(x)]; p\} = \int_0^\infty x^{p-1}\{a[\chi_{1L}(x), \chi_{1U}(x)] + b[\chi_{2L}(x), \chi_{2U}(x)]\}dx = a\int_0^\infty x^{p-1}[\chi_{1L}(x), \chi_{1U}(x)]dx +$
$b\int_0^\infty x^{p-1}[\chi_{2L}(x), \chi_{2U}(x)]dx = a\mathcal{M}_I\{[\chi_{1L}(x), \chi_{1U}(x)]; p\} + b\mathcal{M}_I\{[\chi_{2L}(x), \chi_{2U}(x)]; p\}$.

Appendix 3: Proof of Property 2

By the definition of the interval Mellin transform, we have:
$\mathcal{M}_I\{[\chi_L(cx), \chi_U(cx)]; p\} = \left[\int_0^\infty x^{p-1}\chi_L(cx)\,dx, \int_0^\infty x^{p-1}\chi_U(cx)\,dx\right]$.

Consider the lower endpoint integral. Using the change of variable $z = cx$ (with $c > 0$), we obtain $x = \frac{z}{c}$ and $dx = \frac{1}{c}\,dz$. Hence: $\int_0^\infty x^{p-1}\chi_L(cx)\,dx = \int_0^\infty \left(\frac{z}{c}\right)^{p-1}\chi_L(z)\,\frac{dz}{c} = \frac{1}{c^p}\int_0^\infty z^{p-1}\chi_L(z)\,dz.$

Similarly, for the upper endpoint, we have $\int_0^\infty x^{p-1}\chi_U(cx)\,dx = \frac{1}{c^p}\int_0^\infty z^{p-1}\chi_U(z)\,dz$. Therefore: $\mathcal{M}_I\{[\chi_L(cx),\chi_U(cx)];p\} = \frac{1}{c^p}\left[\int_0^\infty z^{p-1}\chi_L(z)\,dz, \int_0^\infty z^{p-1}\chi_U(z)\,dz\right]$. Recognizing the interval Mellin transform of $[\chi_L,\chi_U]$, we obtain $\mathcal{M}_I\{[\chi_L(cx),\chi_U(cx)];p\} = \frac{1}{c^p}\mathcal{M}_I\{[\chi_L(x),\chi_U(x)];p\}$.

Since $c>0$ and $Re(p)>0$, the kernel x^{p-1} remains non-negative, ensuring preservation of interval ordering.

Appendix 4: Proof of Property 3

From the definition of Mellin transform, we have:

$$\mathcal{M}_I\{x^c[\chi_L(x),\chi_U(x)];p\} = \int_0^\infty x^c[\chi_L(x),\chi_U(x)]x^{p-1}dx = \int_0^\infty x^{c+p-1}[\chi_L(x),\chi_U(x)]dx$$
$$= \Bigg[M\{\chi_L(x);p+c\}, M\{\chi_U(x);p+c\}\Bigg] = [\tilde{\chi_L}(p+c),\tilde{\chi_U}(p+c)].$$

Appendix 5: Proof of Property 4

From the definition of Mellin transform, we have:

$$\mathcal{M}_I\{[\chi_L(x^c),\chi_U(x^c)];p\} = \int_0^\infty[\chi_L(x^c),\chi_U(x^c)]x^{p-1}dx = \Bigg[\int_0^\infty x^{p-1}\chi_L(x^c)dx, \int_0^\infty x^{p-1}\chi_U(x^c)dx\Bigg].$$

Let us take the substitution $x^c = z$. Then, $x = z^{\frac{1}{c}}$ and $dx = \frac{1}{c}z^{\frac{1}{c}-1}dz$. Using the above substitution, we have:

$$\mathcal{M}_I\{[\chi_L(x^c),\chi_U(x^c)];p\} = \Bigg[\int_0^\infty z^{\frac{p-1}{c}}\chi_L(z)\frac{1}{z^{\frac{c-1}{c}}}dz, \int_0^\infty z^{\frac{p-1}{c}}\chi_U(z)\frac{1}{z^{\frac{c-1}{c}}}dz\Bigg]$$
$$= \Bigg[\frac{1}{c}\int_0^\infty z^{\frac{p-1-c+1}{c}}\chi_L(z)dz, \frac{1}{c}\int_0^\infty z^{\frac{p-1-c+1}{c}}\chi_U(z)dz\Bigg] = \Bigg[\frac{1}{c}\int_0^\infty z^{\frac{p}{c}-1}\chi_L(z)dz, \frac{1}{c}\int_0^\infty z^{\frac{p}{c}-1}\chi_U(z)dz\Bigg]$$
$$= \frac{1}{c}M_1\{[\chi_L(x),\chi_U(x)];\tfrac{p}{c}\} = \frac{1}{c}[\tilde{\chi_L}(\tfrac{p}{c}),\tilde{\chi_U}(\tfrac{p}{c})].$$

Appendix 6: Proof of Theorem 2

Case 1: Suppose $[\chi_L,\chi_U]'(x) = [\chi_L'(x),\chi_U'(x)]$. Then $M_1\{[\chi_L,\chi_U]'(x);p\}$ $= \left[\int_0^\infty x^{p-1}\chi_L'(x)\,dx, \int_0^\infty x^{p-1}\chi_U'(x)\,dx\right]$. Using integration by parts and the assumed boundary conditions, we have $\int_0^\infty x^{p-1}\chi_L'(x)\,dx = -(p-1)\int_0^\infty x^{p-2}\chi_L(x)\,dx$, and similarly for $\chi_U(x)$.

Hence $\mathcal{M}_I\{[\chi_L,\chi_U]'(x);p\} = -(p-1)\mathcal{M}_I\{[\chi_L,\chi_U](x);p-1\}$.

Case 2: Suppose $[\chi_L,\chi_U]'(x) = [\chi_U'(x),\chi_L'(x)]$. Proceeding analogously yields: $\mathcal{M}_I\{[\chi_L,\chi_U]'(x);p\} = -(p-1)\mathcal{M}_I\{[\chi_U,\chi_L](x);p-1\}$.

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Conflicts of Interest

The authors declare no conflicts of interest.

References

[1] Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a generalization of sumudu and laplace transform for solving differential equations. *arXiv preprint arXiv:1904.11370*. https://doi.org/10.48550/arXiv.1904.11370.

[2] Zayed, A. I. (2002). A class of fractional integral transforms: A generalization of the fractional fourier transform. *IEEE Transactions on Signal Processing*, *50*(3), 619–627. https://doi.org/10.1109/78.984750.
[3] Luchko, Y. (2019). Some schemata for applications of the integral transforms of mathematical physics. *Mathematics*, *7*(3), 254. https://doi.org/10.3390/math7030254.
[4] Bybliv, O. Y., & Lenyuk, M. P. (1993). Hankel integral transforms of the first kind for piecewise-homogeneous segments with application to problems of mathematical physics. *Journal of Soviet Mathematics*, *66*, 2466–2473. https://doi.org/10.1007/BF01098762.
[5] Watugala, G. K. (1998). Sumudu transform-a new integral transform to solve differential equations and control engineering problems. *Mathematical Engineering in Industry*, *6*(4), 319–329. https://doi.org/10.1080/0020739930240105.
[6] Khan, Z. H., & Khan, W. A. (2008). N-transform-properties and applications. *NUST Journal of Engineering Sciences*, *1*(1), 127–133.
[7] Elzaki, T. M. (2011). The new integral transform elzaki transform. *Global Journal of Pure and Applied Mathematics*, *7*(1), 57–64.
[8] Upadhyaya, L. M. (2022). Introducing the upadhyaya integral transform. *Bulletin of Pure and Applied Sciences*. https://doi.org/10.5958/2320-3226.2019.00051.1.
[9] Jafari, H. (2021). A new general integral transform for solving integral equations. *Journal of Advanced Research*, *32*, 133–138. https://doi.org/10.1016/j.jare.2020.08.016.
[10] Kilbas, A. A., Luchko, Y. F., Martinez, H., & Trujillo, J. J. (2010). Fractional fourier transform in the framework of fractional calculus operators. *Integral Transforms and Special Functions*, *21*(10), 779–795. https://doi.org/10.1080/10652461003676099.
[11] Romero, L., Cerutti, R., & Luque, L. (2011). A new fractional fourier transform and convolutions products. *International Journal of Pure and Applied Mathematics*, *66*(4), 397–408.
[12] Kumar, V. (2020). On a generalized fractional fourier transform. *Palestine Journal of Mathematics*, *9*(2), 903–907.
[13] Mahor, T. C., Mishra, R., & Jain, R. (2020). Fractionalization of fourier sine and fourier cosine transforms and their applications. *Identity*, *14*, 3.
[14] Titchmarsh, E. C. (1937). *Introduction to the theory of fourier integral*. Clarendon Press.
[15] Elfving, G. (1981). The history of mathematics in finland, 1828–1918.
[16] Sun, X., & Yang, Z. (2016). A mellin transform method for solving fuzzy differential equations. *Advances in Difference Equations*, *2016*, 1–12. https://doi.org/10.1186/s13662-016-1027-8.
[17] Azhar, N., & Iqbal, S. (2022). Solution of fuzzy fractional order differential equations by fractional mellin transform method. *Journal of Computational and Applied Mathematics*, *400*, 113727. https://doi.org/10.1016/j.cam.2021.113727.
[18] Singh, P., Gazi, K. H., Rahaman, M., Basuri, T., & Mondal, S. P. (2024). Solution strategy and associated result for fuzzy mellin transformation. *Franklin Open*, 100112. https://doi.org/10.1016/j.fraope.2024.100112.
[19] Jafariani, H., & Tabatabaee, H. (2014). Retinal identification system using fourier-mellin transform and fuzzy clustering. *Indian Journal of Science and Technology*, *7*(9), 1289–1296.
[20] Gong, Y., Xiang, L., Yang, S., & Ma, H. (2020). A new method for ranking interval type-2 fuzzy numbers based on mellin transform. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *28*(04), 591–611. https://doi.org/10.1142/S0218488520500257.
[21] Chen, C.-Y., & Huang, J.-J. (2022). Defuzzify imprecise numbers using the mellin transform and the trade-off between the mean and spread. *Algorithms*, *15*(10), 355. https://doi.org/10.3390/a15100355.

[22] Tamilarasi, G., & Paulraj, S. (2022). An improved solution for the neutrosophic linear programming problems based on mellin's transform. *Soft Computing*, *26*(17), 8497–8507. https://doi.org/10.1007/s00500-022-07252-z.

[23] De, A., Das, S., & Kar, S. (2021). Ranking of interval type 2 fuzzy numbers using correlation coefficient and mellin transform. *Opsearch*, *58*, 1018–1048. https://doi.org/10.1007/s12597-020-00504-2.

[24] Moore, R. E., Kearfott, R. B., & Cloud, M. J. (2009). *Introduction to interval analysis*. SIAM.

[25] Liu, G., & Kreinovich, V. (2010). Fast convolution and fast fourier transform under interval and fuzzy uncertainty. *Journal of Computer and System Sciences*, *76*, 63–76. https://doi.org/10.1016/j.jcss.2009.05.006.

[26] Das, S., Rahman, M. S., Shaikh, A. A., Bhunia, A. K., & Konstantaras, I. (2023). Interval laplace transform and its application in production inventory. *Mathematical Methods in the Applied Sciences*, *46*, 3983–4002. https://doi.org/10.1002/mma.8734.